The Nuclear Mean Field and Its Symmetries

W. Udo Schröder, 2012
Some Basic Questions

- Does a **mean field**, in which all nucleons move independently, make sense at all, given the complexity of the NN interaction?
  - **YES! →**

- Many nuclei are nearly spherical (3D) and have zero or small spins/magnetic moments.

- Mean free path is long for nucleons in nuclear matter.

- Collectivity is not universal, different in different ranges of chart of nuclides.
  - **But →**

- Interior (bulk) and surface have different effective forces acting on a nucleon.

- Finally, comparisons of model predictions to experimental data provides supporting evidence.

**Mean field concept:**
Nucleons in nuclear medium see superposition (→average) of many nearest-neighbor interactions
Fermi gas kinetic energy estimates:

\[ \varepsilon_{F,n} = \frac{3^{2/3} \hbar^2 \pi^{4/3}}{2m_n} \rho_n^{2/3} \quad ; \quad \varepsilon_{F,p} = \tilde{V} + \frac{3^{2/3} \hbar^2 \pi^{4/3}}{2m_p} \rho_p^{2/3} \]

\[ \tilde{V} = 50 \text{ MeV} \left( N - Z \right)/A \approx 11 \text{ MeV} \]

For stable nuclei, Fermi energies for protons and neutrons are equal. Otherwise beta decay.

Light nuclei, \( N \approx Z \approx A/2 \):

\[ \rho_n \approx \rho_p \approx 0.085 \, \text{fm}^{-1} \rightarrow \varepsilon_F \approx 38 \, \text{MeV} \]

Separation energies

\[ S_n (N, Z) = B(N, Z) - B(N - 1, Z) \approx 8 \, \text{MeV} \]

\[ S_p (N, Z) = B(N, Z) - B(N, Z - 1) \approx 8 \, \text{MeV} \]

Potential depth

\[ V_n \approx \varepsilon_{F,n} + S_n \approx 46 \, \text{MeV} \]

Protons somewhat less bound but confined by Coulomb barrier.
The Nuclear Mean Field

General properties of mean potential of nucleus (A=N+Z) with radius $R_A$

- Nucleons close to center ($r=0$): $V(r\approx0) \approx V_0=$const.
- NN forces are short range: $V(r)\rightarrow0$ for $r > R_A$ rapidly but range of potential is $R > R_A$
- NN forces have saturation character: central mass density $\rho(r\approx0) \approx$ const. for all $A$, $V_0 = $ const.
- Total s.p. interaction: $H = H_{\text{nucl}} + H_{\text{elm}} + H_{\text{weak}} + \ldots$
  elm=electromagnetic, not all conservative!

**Trial potentials**

**Infinite Square Well**

$$V(r) = \begin{cases} 0 & r \leq R \\ +\infty & r > R \end{cases} \quad \text{"box"}$$

**Finite Square Well**

$$V(r) = \begin{cases} -V_0 & r \leq R \\ 0 & r > R \end{cases}$$

**Harmonic Oscillator**

$$V(r) = \left(\frac{M}{2\omega_0^2}\right) \cdot r^2$$

**Saxon–Woods**

$$V(r) = \frac{-V_0}{1 + \exp\left\{-\frac{r - R}{a}\right\}}$$

- $M=$inertia
- $\omega_0=$frequency
- $V_0 =$ depth
- $a=$diffuseness
Reducing The Nuclear A-Body Schrödinger Problem

Nucleus with A = N+Z; NN potential $V(r_{ij})$

$$\hat{H} = \sum_{i=1}^{A} \left[ \hat{T}_i + \frac{1}{2} \sum_{j \neq i} V(r_{ij}) \right] = 3A\text{-dim. Schrödinger problem}$$

Neglect residual interactions → independent – particle model

Shell Model: $\hat{H} = \sum_{i=1}^{A} \hat{H}_i + \overline{V}_{\text{res}} \approx \sum_{i=1}^{A} \hat{H}_i$  
Single – particle Hamiltonian: $\hat{H}_i = \hat{T}_i + \overline{V}(\bar{r}_i)$

$$\begin{cases} -\hbar^2 \\ 2m_i \end{cases} \Delta_i + \overline{V}(\bar{r}_i) \psi_i(\bar{r}_i) = \varepsilon_i \cdot \psi_i(\bar{r}_i)$$

3-dimensional Schrödinger problem

$A$ – body wave function: $\Psi(\bar{r}_1, \ldots, \bar{r}_A) = \det |\psi_i(\bar{r}_k)|$ antisymmetric

Total energy: $E = \sum_i \varepsilon_i$

Symmetries $\rightarrow$ Further simplifications of 3D Schrödinger problem
Single-Particle Symmetries

3D \to 1D Schrödinger problem

\[ \begin{cases} -\hbar^2 & \Delta + \bar{V} (\vec{r}) \psi_n (\vec{r}) = \varepsilon_n \cdot \psi_n (\vec{r}) \\ \frac{\hbar^2}{2m} & \end{cases} \]

Mean field \( \bar{V} (\vec{r}) \) confines system \( \rightarrow \) discrete energy spectrum
main quantum numbers \( n = 1, 2... \)

If central potential : \( \bar{V} (\vec{r}) = V (r) \rightarrow \) no angle dependent torques \( \rightarrow \ell \) conserved

Spherical symmetry, rotational invariance, decoupled radial and angular motion

Product wave functions \( \psi_{n\ell} (\vec{r}) = R_{n\ell} (r) \cdot Y (\ell, \theta, \phi) \cdot \chi_{\text{spin}} = \frac{u_{n\ell} (r)}{r} \cdot Y_m (\theta, \phi) \cdot \chi_{\text{spin}} \)

Integrability : \( \int_0^\infty |R_{n\ell} (r)|^2 \cdot r^2 dr < \infty \rightarrow \int_0^\infty |u_{n\ell} (r)|^2 dr < \infty \) (finite)

Spherical harmonics = eigen functions to angular momentum & parity operators

\[ \hat{L}^2 Y_m (\theta, \phi) = \ell (\ell + 1) \cdot \hbar^2 Y_m (\theta, \phi) \]

\[ \hat{L}_z Y_m (\theta, \phi) = m \cdot \hbar \cdot Y_m (\theta, \phi) \]

\[ \hat{\Pi} \psi_{n\ell} (\vec{r}) := \psi_{n\ell} (-\vec{r}) \rightarrow \hat{\Pi} Y_m (\theta, \phi) = (-1)\ell Y_m (\theta, \phi) \]

\[ [\hat{\mathcal{H}}, \hat{\mathcal{L}}] = [\hat{\mathcal{H}}, \hat{\mathcal{L}}_z] = [\hat{\mathcal{H}}, \hat{\Pi}] = 0 \]

(Treat spin \( \chi_{\text{spin}} \) later)

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Charge Independence: Nuclear Iso(baric)Spin

Distinguish 2 similar nucleon states (proton, neutron) by isospin $\bar{t}$

“isospin $t_z$ up” = proton or “isospin $t_z$ down” = neutron $\rightarrow t = 1/2$

Use formalism identical to spin-1/2 formalism.

*Nucleon basis wave functions* : $\psi(\vec{r},\vec{s},\bar{t}) = \varphi(\vec{r}) \cdot \chi_s(\vec{s}) \cdot \zeta_{\bar{t}}(\bar{t})$

*Adopt 2 isospin basis vectors*

Proton : $\zeta_{\bar{t}=p}$ $\rightarrow |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\rightarrow \hat{t}_z |0\rangle = \frac{1}{2} \cdot |0\rangle$

Neutron : $\zeta_{\bar{t}=n}$ $\rightarrow |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow \hat{t}_z |1\rangle = -\frac{1}{2} \cdot |1\rangle$

*Matrix representation of isospin operators in this basis*

$\hat{t}_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \tau_x$  $\hat{t}_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2} \tau_y$  $\hat{t}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \tau_z$

$\tau_i$ analogous to Pauli spin matrices

*Ladder operators* : $\hat{t}_+ := \hat{t}_x + i \cdot \hat{t}_z$  $\hat{t}_- := \hat{t}_x - i \cdot \hat{t}_z$

$\rightarrow \hat{t}_+ |n\rangle = |p\rangle$  and  $\hat{t}_- |p\rangle = |n\rangle$

$\{\hat{t}_+, \hat{t}_z, \hat{t}_-\}$ : sphereical tensor of rank 1
Isospin Operators

Projection operators defined for nucleon wave function \( \Xi \)

Proton projector: \( \hat{P}_p := \frac{1}{2}(\hat{1} + \hat{z}) \); neutron projector: \( \hat{P}_n := \frac{1}{2}(\hat{1} - \hat{z}) \)

\( \rightarrow \) N charge projector: \( \hat{q} := (e/2)(\hat{1} + \hat{z}) = \text{Example of isovector operator} \)

Charge independence of NN forces \( \rightarrow \)
A-nucleon energy should depend strongly on nucleonic configuration, not on # of n vs. # p (Coulomb energy breaks symmetry).
\( \rightarrow \) Define total isospin additional quantum number for A-nucleon state.

Total isospin vector operator: \( \hat{T} = \frac{1}{2} \sum_{i=1}^{A} \hat{\tau}_i \); \( \hat{T}_z = \frac{1}{2} \sum_{i=1}^{A} \hat{\tau}_{i,z} \) defined for isospin function \( \Xi_T \)

similar to spin & spin projection operators

\( \hat{T}^2 \Xi_T = T(T+1) \Xi_T \); \( \hat{T}_z \Xi_T = \frac{1}{2}(Z-N) \Xi_T \)  \( \text{0} \leq T \leq A/2 \); \(-T \leq T_z \leq +T \)

low total \( T \) \( \rightarrow \) more antisymmetric isospin wave function
low total \( S \) \( \rightarrow \) more antisymmetric spin wave function

Expect total isospin symmetry to be “good” for light nuclei (weak Coulomb) but problematic for heavy nuclei (high Z). Strong Coulomb force mixes states with different isospins (\( T \)) \( \rightarrow \) “breaks symmetry.”
2-Particle Isospin Coupling

Use spin/angular momentum formalism: \( t \to (2t+1) \) iso-projections

\[
\begin{pmatrix}
  t_1 \\
  m_{t_1}
\end{pmatrix}
\begin{pmatrix}
  t_2 \\
  m_{t_2}
\end{pmatrix}
:=
\begin{pmatrix}
  t_1 & t_2 \\
  m_{t_1} & m_{t_2}
\end{pmatrix}
\text{can couple to} \ |t_1 - t_2| \leq T \leq |t_1 + t_2|
\]

Total isospin states \( \Xi_{T,M_T} \to \begin{pmatrix} T \\ M_T \end{pmatrix} = \sum_{m_{t_1},m_{t_2}} \begin{pmatrix} t_1 & t_2 \\ m_{t_1} & m_{t_2} \end{pmatrix} \begin{pmatrix} T \\ M_T \end{pmatrix} \begin{pmatrix} t_1 & t_2 \\ m_{t_1} & m_{t_2} \end{pmatrix} \)

Iso – antisymmetric: \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) Iso – symmetric: \( \begin{pmatrix} T = 1 \\ M_T \end{pmatrix} \) \((M_T = -1, 0, 1)\)

\[
\Psi \left[ \left[ j_1 j_2 \right]_M^J \right]_{M_T}^T \left[ \Psi_{j_1m_1} (1) \Psi_{j_2m_2} (2) + (-1)^T \Psi_{j_1m_1} (2) \Psi_{j_2m_2} (1) \right] \cdot \Xi_{T,M_T}
\]

Totally antisymmetric spin-isospin component
Possible isospin assignments for heavier nuclei:

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$^{45}_{20}Ca$</th>
<th>$^{45}_{21}Sc$</th>
<th>$^{45}_{22}Ti$</th>
<th>$^{45}_{23}V$</th>
<th>$^{45}_{24}Cr$</th>
<th>$^{45}_{25}Mn$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_z$</td>
<td>-5/2</td>
<td>-3/2</td>
<td>-1/2</td>
<td>+1/2</td>
<td>+3/2</td>
<td>+5/2</td>
</tr>
<tr>
<td>$T$</td>
<td>$\geq 5/2$</td>
<td>$\geq 3/2$</td>
<td>$\geq 1/2$</td>
<td>$\geq 1/2$</td>
<td>$\geq 3/2$</td>
<td>$\geq 5/2$</td>
</tr>
</tbody>
</table>
Example: A=12 Isospin Triplet

$^{12}\text{B}$ and $^{12}\text{N}$ isospin T≥1, mostly T=1

$^{12}\text{C}$ isospin T≥0, mostly T=0 (most tightly bound g.s. and near g.s. levels).
T=1 expected at higher E* (likely 15.11 MeV), consistent with decay strengths.

Isobaric Analog States: Mirror Nuclei

For same $T$, wfs for protons and neutrons are similar → “Mirror Nuclei” $^{19}$F & $^{19}$Ne. Ne has higher Coulomb energy. Estimate: $\Delta E_C \approx 50$keV, not counting some trivial differences ($m_n \neq m_p$).

$$E_C = \frac{3}{5} \frac{e^2 Z^2}{r_0 A^{1/3}} \rightarrow \Delta E_C \approx E_C \cdot \left(\frac{2}{Z}\right)$$
Isobaric Analog States: Light Nuclei

Lower levels of $A=14$ isobars, ground states of $^{14}\text{C}$ and $^{14}\text{O}$ have been shifted relative to $^{14}\text{N}$ to account for the difference in $n-p$ masses and $E_C$.

Energy levels in $^{14}\text{C}$ and $^{14}\text{O}$ have to be $T=1$. Levels in $^{14}\text{N}$ are mostly $T=0$, but there are $T=1$ levels as well ($E^*=2.31, 8.06 \text{ MeV}$)

How does one know??

→ Look at decay and reactions

$^{12}\text{C}(\alpha, \text{d})^{14}\text{N}$ populates $T=1 - ^{14}\text{N}$ states only weekly ($10^{-3}$) compared to $T=0$ states. Isospin forbidden! → Selection rules

Isospin Selection Rules

Charge independence of nuclear interactions → Isospin is conserved in nuclear reactions! Other types are strongly suppressed.

Nuclear $\beta$ decay (n $\leftrightarrow$ p) connect states of same $T$ $\rightarrow$ $\Delta T=0$ (as observed!)

Nuclear $\gamma$ decay mediated by isovector transition $\rightarrow$ $\Delta T=0,1$ for pure $T$ states

Example: electric s.p. dipole operator $\hat{D}_{E1} = e \cdot z = e \cdot |\vec{r} - \vec{R}_{cm}| \cdot \cos \theta$

$R_{cm} =$ position of center of nucleus (c.o.m.)

Iso-vector electric transition operator (acts only on charged particles : p), e.g., $E1$

$\hat{\mathcal{O}}_{E1} = \sum_p (\vec{r}_p - \vec{R}_{cm})_z = \sum_i (e/2)(\hat{\mathbf{\hat{r}}}_i z_i)(\vec{r}_i - \vec{R}_{cm})_z = (e/2)\sum_i (\vec{r}_i - \vec{R}_{cm})_z + (e/2)\sum_i \hat{r}_i z_i$  $\rightarrow \Delta T=0$

Elm. transitions:

$\Delta T = 0,1$  $\Delta T_z = 0$  $T = 0 \times \n T = 0$

Beta decay and charge exchange reactions change n $\leftrightarrow$ p, $\rightarrow \Delta T=1$
Isospin Selection Rules for Gamma Decay

\[ \hat{O}_{E1} = \text{iso – vector op} \]
\[ \rightarrow \Delta T = 0 \text{ E1 inhibited} \]
\[ \Gamma(E1) \ll \Gamma_{sp}(E1) \]

\[ \Gamma(M1) \sim 10^{-2} \Gamma_{sp}(M1) \]
\[ T = 0 \rightarrow T = 0 \text{ M1 inhibited} \]
\[ \text{compared to } \Delta T = 1 \]
Isobaric Analog States in Compound Nuclear Reactions

\[ \begin{align*}
E^* &= 7.8 \text{ MeV} \\
E^* &= 0 \\
\Delta E_C &= \frac{3}{5} e^2 (2Z - 1)/R \\
\end{align*} \]

Experimental (p, n) neutron spectrum

\[ \begin{align*}
dN/\text{d}E_n &< 7.8 \text{ MeV} > \\
T_\text{IAS} &= \frac{11}{2} \\
T_\text{Background} &= \frac{9}{2} \\
E_n &\rightarrow
\end{align*} \]
Questions

1. Give a rationale for the quadratic radial dependence $V_{\text{coil}}(r)$ of the Coulomb potential in the interior of a homogeneously charged nucleus.

2. Write down an approximate relation between the mean-field oscillator strength and the mean-square radius $<r^2>^{1/2}$ of the nuclear matter distribution.

3. Consider 3 wave functions $\psi_i(x_n)$ for 3 fermions (n=1,2,3). Out of the $\psi_i(x_n)$ construct a fully antisymmetric 3-particle wave function.

4. Show the validity of the commutation relation $\left[\hat{t}_x, \hat{t}_y\right] = -i \cdot \hat{t}_z$

5. If NN forces are charge independent, why do T=0 and T=1 deuteron states have different energies?

6. Why is isospin not a good symmetry for heavy nuclei?

7. What is the difference in state energies of good isospin multiplets? Calculate the Coulomb contribution to this difference.

8. The ground state of $^{81}\text{Br}$ (Z=35) has a spin of 3/2$. Give the isospin quantum numbers $(T, T_z)$ of its isobaric analog state in $^{81}\text{Kr}(Z=36)$. Approximately at what excitation energy in $^{81}\text{Br}$ does one expect the IAS of the $^{81}\text{Kr}$
That’s It (for now)