Nuclear Gamma Decay
Protons in nuclei = moving charges

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Transversal E, M Fields
Photon Spin 1

Point Charges
Nuclear Charge Distribution

Monopole $\ell = 0$

Dipole $\ell = 1$

Quadrupole $\ell = 2$

$\text{t-Dependent Electromagnetic Fields}$
Photons

Photons: generated by moving charge distributions. Distributions can have intrinsic angular momentum (spin)

Rotations: move charge distributions and direction of spin

Spherical tensor \( \vec{u}_{\pm 1} = \frac{\mp 1}{\sqrt{2}} (\vec{u}_x \pm i \vec{u}_y) \), \( \vec{u}_0 \)

(\( \vec{u}_i \) spatial unit vectors = "polarization")

\[
\vec{A}(\vec{r}, t) = \vec{u}_{\pm 1} \exp \left( i \vec{k} \cdot \vec{r} - i \vec{\omega} \cdot t \right) = \text{Vector field rotating about z axis}
\]

\[
= \frac{-1}{\sqrt{2}} \left\{ \vec{u}_x \cos(\vec{k} \cdot \vec{r} - \vec{\omega} t) - \vec{u}_y \sin(\vec{k} \cdot \vec{r} - \vec{\omega} t) \right\} - \frac{-i}{\sqrt{2}} \left\{ \vec{u}_x \sin(\vec{k} \cdot \vec{r} - \vec{\omega} t) + \vec{u}_y \cos(\vec{k} \cdot \vec{r} - \vec{\omega} t) \right\}
\]

Analogous for \( S_x, S_y \)

Intrinsic spin=1 system

Have to couple S with orbital \( L \rightarrow J \)
Electromagnetic Multipole Radiation

Following Greiner & Maruhn (“Nuclear Models”), Jelley (“Fundamentals of Nuclear Physics”)

Scalar photon wave equation: \[ (\Delta + k^2) \Phi^\lambda_\mu(\vec{r}) = 0 \] Helmholtz Equ.

Transverse solutions \( \vec{\nabla} \cdot \vec{A} = 0 \) only

\[ \Phi^\lambda_\mu(\vec{r}) = j^\lambda(kr) \cdot Y^\lambda_\mu(\hat{\vec{r}}) \quad k : \omega/c \text{ spherical Bessel functions } j^\lambda(kr) \]

\( \Phi^\lambda_\mu \to (\text{orbital}) \) angular momentum \( \lambda \), projection \( \mu \)

Helmholtz Eq. commutes with angular momentum operator \( \hat{\mathbf{L}} \to \) construct orthogonal solutions with well defined angular momentum \( \to \) vector potential:

\[ \vec{A}^\ell_m(\vec{r}; M) = \frac{1}{\hbar \sqrt{\ell (\ell + 1)}} \hat{\mathbf{L}} j^\ell(kr) \cdot Y^\ell_m(\hat{\vec{r}}) \quad \text{magnetic } \ell - \text{pole, } \pi^\ell = (-1)^{\ell+1} \]

\[ \vec{A}^\ell_m(\vec{r}; E) = \frac{-i}{\hbar k \sqrt{\ell (\ell + 1)}} \nabla \times \hat{\mathbf{L}} j^\ell(kr) \cdot Y^\ell_m(\hat{\vec{r}}) \quad \text{electric } \ell - \text{pole, } \pi^\ell = (-1)^\ell \]

Transversality: \( \nabla \cdot \vec{A}^\ell_m(\vec{r}; M) \propto \nabla \cdot \hat{\mathbf{L}} \propto \nabla \cdot (\vec{r} \times \nabla) = 0 \)

\( \nabla \cdot \vec{A}^\ell_m(\vec{r}; E) \propto \nabla \cdot \nabla \times \hat{\mathbf{L}} = 0 \)
Interaction with Radiation Field

Current of moving nuclear charges $\hat{j}(\vec{r}) \rightarrow$ interaction

$$\hat{H}_{\text{int}} = -\frac{1}{c_{\text{Nucleus}}} \int d^3\vec{r} \left[ \hat{j}(\vec{r}) \right] \cdot \hat{A}_{\text{Photon}}(\vec{r}, t)$$

For plane wave in volume $V$:

$$\hat{A}_{\text{Photon}}(\vec{r}, t) = \vec{u} \sqrt{\frac{2\pi \hbar c^2}{\omega V}} (ae^{i(\vec{k} \cdot \vec{r} \omega t}) + a^* e^{-i(\vec{k} \cdot \vec{r} \omega t)})$$

Expressed plane elm wave (= Wf of photon) in terms of multi-pole components, each of well defined (“good”) angular momentum

$$\vec{u}_m e^{ikz} = -m\sqrt{2} \sum_\ell \sqrt{2\ell + 1} i^\ell \left( \tilde{A}_m^\ell(\vec{r}; M) + im\tilde{A}_m^\ell(\vec{r}; E) \right) \quad m = \pm 1$$

Matrix element:

$$M_{\alpha\beta}(k, m) = -m\sqrt{2} \sum_\ell \sqrt{2\ell + 1} i^\ell \int d^3\vec{r} \langle \alpha | \hat{j} | \beta \rangle \left( \tilde{A}_m^\ell(\vec{r}; M) + im\tilde{A}_m^\ell(\vec{r}; E) \right)$$

Initial and final nuclear WF
Electric and Magnetic Multipole ME

\[ M_{\alpha\beta}(k, m; E\ell) = -\sqrt{2\pi} \sqrt{2\ell + 1} i^{\ell + 1} \int_{\text{Nucleus}} d^3\vec{r} \langle \alpha | \hat{\vec{j}} | \beta \rangle \cdot \vec{A}_m(\vec{r}; E) \]

\[ M_{\alpha\beta}(k, m; M\ell) = -m\sqrt{2\pi} \sqrt{2\ell + 1} i^{\ell} \int_{\text{Nucleus}} d^3\vec{r} \langle \alpha | \hat{\vec{j}} | \beta \rangle \cdot \vec{A}_m(\vec{r}; M) \]

\[ \gamma \text{ Transition} \]

\[ E_\alpha - E_\beta = \hbar \omega_{\alpha\beta} = c\hbar k \]

\[ \left| I_\alpha - I_\beta \right| \leq \ell \leq \left| I_\alpha + I_\beta \right| \]

\[ \pi_\alpha = \begin{cases} (-1)^\ell \pi_\beta & E\ell \\ (-1)^{\ell+1} \pi_\beta & M\ell \end{cases} \]
### Long Wave Length Limit

**Basically true for all γ spectroscopy:**

Low – energy limit: \( k_\gamma \cdot R_{\text{Nucleus}} \ll 1, \quad R_{\text{Nucleus}} \leq 10 \text{ fm} \)

\[
E_\gamma = (\hbar c) \cdot k_\gamma \approx (200 \text{ MeV fm}) \cdot k_\gamma \ll \frac{200 \text{ MeV fm}}{10 \text{ fm}} = 20 \text{ MeV}
\]

\[ j_\ell(kr) \approx \frac{(kr)^\ell}{(2\ell + 1)!!} \quad \text{for} \quad kr \ll 1 \rightarrow j_{\ell+1}(kr) \ll j_{\ell-1}(kr) \quad (k := k_\gamma)
\]

\[ \bar{A}_m^\ell(\vec{r}; E) \approx \sqrt{\frac{\ell+1}{\ell}} \frac{1}{k} \vec{\nabla} \left[ j_\ell(kr) \cdot Y_m^\ell(\hat{r}) \right]
\]

\[ M_{\beta\alpha}(k, m; E^\ell) \approx \sqrt{2\pi i} \frac{i^{\ell+1}}{k} \sqrt{\frac{(\ell+1)(2\ell+1)}{\ell}} \int d^3\vec{r} \left[ \vec{\nabla} \cdot \langle \beta | \hat{j} | \alpha \rangle \right] \cdot j_\ell(kr)Y_m^\ell(\hat{r})
\]

\[ 0 = \frac{\partial}{\partial t} \langle \beta(t) | \rho(\vec{r}) | \alpha(t) \rangle + \vec{\nabla} \cdot \langle \beta(t) | \hat{j}(\vec{r}) | \alpha(t) \rangle = \quad \text{Continuity Equ.}
\]

\[ = -\frac{i}{\hbar} \left( E_\alpha - E_\beta \right) \langle \beta(t) | \rho(\vec{r}) | \alpha(t) \rangle + \vec{\nabla} \cdot \langle \beta(t) | \hat{j}(\vec{r}) | \alpha(t) \rangle
\]
**Transition Rate**

\[
M_{\beta\alpha}(k, m; E\ell) \approx \sqrt{2\pi i^\ell c} \sqrt{\frac{(\ell + 1)(2\ell + 1)}{\ell}} \int d^3\vec{r} \langle \beta | \hat{\rho}(\vec{r}) | \alpha \rangle j_\ell(kr) Y_m(\hat{r})
\]

Long – wave limit : \( j_\ell(kr) \approx (kr)^\ell/(2\ell + 1)!! \) →

El. nuclear ME \( \langle \beta | \hat{\Omega}_{\ell m}(E) | \alpha \rangle := \int d^3\vec{r} \langle \beta | \hat{\rho}(\vec{r}) | \alpha \rangle r^\ell Y_m(\hat{r}) \)

Magn. nuclear ME \( \langle \beta | \hat{\Omega}_{\ell m}(M) | \alpha \rangle := \frac{-1}{c(\ell + 1)} \int d^3\vec{r} \langle \beta | \hat{j}(\vec{r}) | \alpha \rangle \cdot \hat{L} r^\ell Y_m(\hat{r}) \)

Nucleus : \( | \alpha(t) \rangle = e^{-iE_{\alpha t}/\hbar} | \alpha(0) \rangle \quad \text{Photon : } | \vec{k}, \mu \rangle = \sqrt{\frac{2\pi\hbar c}{kV}} \hat{u}_\mu e^{i(k \cdot \vec{r} - \omega t)} \)

\[
P_{i \rightarrow f} = \frac{2\pi}{\hbar} \int dE_\gamma \left| \langle f | \hat{H}_{\text{int}} | i \rangle \right|^2 \rho_f(E_\gamma) \quad \rho_f(E_\gamma) = 1_{\text{nucl}} \cdot \frac{dn_\gamma}{dE_\gamma}
\]

Particle in a box : \( V = L^3 \rightarrow dn_\gamma = \frac{V}{(2\pi)^3} k^2 dk d\Omega \)

\( \text{V cancelled by normalization} \)
Reduced Transition MEs

\[ P_{i \rightarrow f}(\ell m; E) = \frac{(\ell + 1)(2\ell + 1)}{\ell (2\ell + 1)!!} \cdot \frac{k^{2\ell + 1}}{\hbar} \left| \langle f | \hat{\Omega}_{\ell m}(E) | i \rangle \right|^2 d\Omega \]

Reduced ME unpolarized proj + target: Average over \( M_i \), sum over \( M_f \)

\[ B(E_{\ell}; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \left| \langle f | \hat{\Omega}_{\ell m}(E) | i \rangle \right|^2 \sum_{M_i, M_f, m} \left( I_i \ell | I_f M_i m \right)_m \right) \]

\[ B(E_{\ell}; I_i \rightarrow I_f) = \frac{2I_f + 1}{2I_i + 1} \left| \langle f | \hat{\Omega}_{\ell m}(E) | i \rangle \right|^2 \]
\[
P_{i \rightarrow f}(E_{\ell}, M_{\ell}) = \frac{8\pi (\ell + 1)(2\ell + 1)}{\ell [ (2\ell + 1)!]^{2}} \cdot \frac{K^{2\ell+1}}{\hbar} B(E_{\ell}, M_{\ell}; I_{i} \rightarrow I_{f})
\]

\[
B(E_{\ell}; I_{i} \rightarrow I_{f}) = \frac{2I_{f} + 1}{2I_{i} + 1} \left| \int d^{3}\vec{r} \left\langle f \right| \hat{\rho}(\vec{r}) r^{\ell} Y^{\ell}(\hat{\vec{r}}) \left| i \right\rangle \right|^{2}
\]

\[
B(M_{\ell}; I_{i} \rightarrow I_{f}) = \frac{2I_{f} + 1}{2I_{i} + 1} \frac{1}{c^{2}(\ell + 1)^{2}} \left| \int d^{3}\vec{r} \left\langle f \right| \hat{J}(\vec{r}) \cdot \hat{L} r^{\ell} Y^{\ell}(\hat{\vec{r}}) \left| i \right\rangle \right|^{2}
\]

**Transition energy dependence:** \( P_{i \rightarrow f} \sim \omega_{if}^{2\ell+1} \) for \( B \approx \text{const.} \)

But \( B(E_{\ell}) \sim R^{2\ell} \rightarrow P_{i \rightarrow f}(E_{\ell}) \sim \omega_{if}(k_{if}R)^{2\ell} \rightarrow \text{large } \ell \text{ suppressed} \)

\( \ll 1 \)

\( B(M_{\ell}) \sim \left( \frac{\hbar}{mcR} \right)^{2} R^{2\ell} \rightarrow P_{i \rightarrow f}(M_{\ell}) \sim \left( \frac{\hbar}{mcR} \right)^{2} \)

\( P_{i \rightarrow f}(E_{\ell}) \sim 10^{-4} P_{i \rightarrow f}(E_{\ell}) \rightarrow \)

\( \rightarrow \text{magnetic multipole transitions } M_{\ell} \text{ suppressed.} \)
Weisskopf’s Estimates of s.p. Transition Probability

Consider single nucleon in circular orbit (extreme SM) and long wave (kR \ll 1) \rightarrow particle wave function

\[ \varphi(\vec{r}) \propto \varphi_{\ell}(r) \cdot Y_m^\ell \]
\[ s - \text{wave} : \varphi_{\ell}(r) \approx \left( \frac{4\pi}{3} R^3 \right)^{-1/2} \]

\[ \langle f \mid \hat{\Omega}_{\ell m}(E) \mid i \rangle \propto e \int dr \, r^2 \varphi_f^*(r) \varphi_i(r) r^\ell \int d\Omega \, Y_{m_i}^{\ell i}^*(\Omega) Y_{m}^{\ell}(\Omega) Y_{m_f}^{\ell i}(\Omega) \]

\[ \langle f \mid \hat{\Omega}_{\ell m}(M) \mid i \rangle \approx \frac{e\hbar}{mc} \frac{1}{R} \cdot \langle r^\ell Y_m^\ell \rangle \]

\[ P(E\ell) \approx \frac{4.4(\ell + 1)}{\ell \left[ (2\ell + 1)!! \right]^2} \left( \frac{3}{\ell + 3} \right)^2 \left( \frac{\hbar \omega}{197 \text{ MeV}} \right)^{2\ell+1} \left( \frac{R}{\text{fm}} \right)^{2\ell} \frac{10^{21}}{\text{s}} \]

\[ P(M\ell) \approx \frac{1.9(\ell + 1)}{\ell \left[ (2\ell + 1)!! \right]^2} \left( \frac{3}{\ell + 2} \right)^2 \left( \frac{\hbar \omega}{197 \text{ MeV}} \right)^{2\ell+1} \left( \frac{R}{\text{fm}} \right)^{2\ell-2} \frac{10^{21}}{\text{s}} \]
Weisskopf’s $E\ell$ Estimates

**single particle:**

$$P(E\ell) \sim E_{\gamma}^{2\ell+1} \cdot A^{2/3}$$

**Experimental E1:**
Factor $10^3$-$10^7$ slower than s.p. WE

- Configurations more complicated than s.p. model,
- time required for rearrangement

**Experimental E2:**
Factor $10^2$ faster than s.p. WE

- Collective states, more than 1 nucleon.

**Experimental $E\ell$ ($\ell>2$):**
App. correctly predicted.

$T_{1/2}$ for solid lines have been corrected for internal conversion.
Weisskopf’s $E_\ell$ Estimates

\[
P(M_\ell) \sim E_\gamma^{2\ell+1} \cdot A^{(\ell-1)2/3}
\]

**single particle**:  

\[
T_{1/2} \text{ for solid lines have been corrected for internal conversion.}
\]

Experimental $M_\ell$:  
Several order weaker than $E_\ell$ transitions.
Special B Values for Rotational Bands

**ee rotational bands** $0^+, 2^+, 4^+, 6^+, 8^+, ...$

$$B(E2, I + 2 \rightarrow I) = \frac{15}{32\pi} e^2 Q_0 \frac{(\ell + 1)(\ell + 2)}{(2I + 3)(2I + 5)}$$

**o - A rotational bands** $I = K, K + 1, K + 2, K + 3, ...$

$$B(E2, I + 1 \rightarrow I) = \frac{15}{16\pi} e^2 Q_0 \frac{K^2 (I + 1 + K)(I + 1 - K)}{I (I + 1)(I + 2)(I + 3)}$$

$$B(E2, I + 2 \rightarrow I) = \frac{15}{32\pi} e^2 Q_0 \frac{(\ell + 1 + K)(\ell + 1 - K)(\ell + 2 + K)(\ell + 2 - K)}{(I + 1)(I + 2)(2I + 3)(2I + 5)}$$

$$B(M1, I + 1 \rightarrow I) = \frac{3}{4\pi} \left( \frac{e\hbar}{2m_nc} \right)^2 (g_\Omega - g_R) \frac{\Omega^2 (I + 1 + K)(I + 1 - K)}{(I + 1)(2I + 3)}$$

$\Omega \rightarrow \text{rot of s.p., } R \rightarrow \text{rot of nuclear core}$

K-selection rules for rot bands: $\Delta K = |K_i - K_f| \leq 1$. K-forbiddenness: $\nu = \Delta K - 1$
Summary: WE sp Estimates/W.u.

\[ P(E1) = 1.59 \times 10^{15} \, E_\gamma^3 \cdot B(E1) \, s^{-1} \]
\[ P(E2) = 1.22 \times 10^9 \, E_\gamma^5 \cdot B(E2) \, s^{-1} \]
\[ P(E3) = 5.67 \times 10^2 \, E_\gamma^7 \cdot B(E3) \, s^{-1} \quad \text{units} \]
\[ P(E4) = 1.69 \times 10^{-4} \, E_\gamma^9 \cdot B(E4) \, s^{-1} \]
\[ \left[ B(E\ell) \right] = e^2 (fm)^{2\ell} \]
\[ P(M1) = 1.76 \times 10^{13} \, E_\gamma^3 \cdot B(M1) \, s^{-1} \]
\[ P(M2) = 1.35 \times 10^7 \, E_\gamma^5 \cdot B(M2) \, s^{-1} \]
\[ P(M3) = 6.28 \times 10^0 \, E_\gamma^7 \cdot B(M3) \, s^{-1} \]
\[ P(M4) = 1.87 \times 10^{-6} \, E_\gamma^9 \cdot B(M4) \, s^{-1} \]

**Weisskopf Units (W.u.)**

\[ \Gamma(E1) = 6.8 \times 10^{-2} \, A^{2/3} \left( E_\gamma / \text{MeV} \right)^3 \text{eV} \]
\[ \Gamma(E2) = 4.9 \times 10^{-8} \, A^{4/3} \left( E_\gamma / \text{MeV} \right)^5 \text{eV} \]
\[ \Gamma(E3) = 2.3 \times 10^{-14} \, A^2 \left( E_\gamma / \text{MeV} \right)^7 \text{eV} \]
\[ \Gamma(M1) = 2.1 \times 10^{-2} \left( E_\gamma / \text{MeV} \right)^3 \text{eV} \]
Collectively Enhanced Transition Rates

\[ \langle E2 \rangle = \langle \psi_f | \hat{\Omega}(E2) | \psi_i \rangle = \left\langle \left[ (f_{7/2})^3 \right]_{7^{-}} \left| \sum_{n=1}^{3} \hat{\Omega}_n(E2) \left[ (f_{7/2})^3 \right]_{5^{-}} \right\rangle \right. \\
\langle E2 \rangle = \sum_{n=1}^{3} \langle \psi_f | \hat{\Omega}_n(E2) | \psi_i \rangle \sim 3 \Omega_{fi}(E2) \rightarrow \Gamma(E2) \leq 9 |\Omega_{fi}(E2)|^2 = 9 \Gamma_{sp}(E2) \\
\Gamma(E2) \leq N^2 \Gamma_{sp}(E2) \text{ because of destructive interference} \]
Isospin Selection Rules

Reason: n/p rearrangements \(\rightarrow\) multipole charge distributions

Photon interacts only with 1 nucleon \((t = 1/2)\) \(\rightarrow\) \(\Delta T = 0, 1\)

Example \(E1\) transitions:

\[
\hat{O}_{E1} = \sum_p e(\vec{r}_p - \vec{R}_{cm}) = \frac{1}{2} \sum_i e t_{z_i} (\vec{r}_i - \vec{R}_{cm}) = \text{iso-vector op}
\]

Enhanced (collective) \(E2, E3\) transitions \(\Delta T = 0\)

Collective (rot or vib) WF does not change in transition.

\[
\hat{O}_{M1} = \frac{1}{2} \sum_i (1 - 2t_{z_i}) \hat{I}_i + \frac{1}{2} \sum_i (1 - 2t_{z_i}) g_p \hat{s}_i + \frac{1}{2} \sum_i (1 + 2t_{z_i}) g_n \hat{s}_i
\]

\[
= \frac{1}{2} \hat{I} + \frac{1}{2} \sum_i (g_n + g_p - 1) \hat{s}_i + \sum_i t_{z_i} \left[ (g_n - g_p) \hat{s}_i - \hat{l}_i \right]
\]

\[
g_n = -3.8 \quad g_p = +5.6
\]
Gamma Decay of Isobaric Analog States

For same $T$, wfs for protons and neutrons are similar
→ “Mirror Nuclei” $^{19}\text{Ne}$ and $^{19}\text{F}$
Isospin Selection Rules for Gamma Decay

\[ \hat{O}_{E1} = \text{iso-vector op} \]
\[ \rightarrow \Delta T = 0 \text{ E1 inhibited} \]
\[ \Gamma(E1) \ll \Gamma_{sp}(E1) \]

\[ \Gamma(M1) \sim 10^{-2} \Gamma_{sp}(M1) \]
\[ T = 0 \rightarrow T = 0 \text{ M1 inhibited} \]
\[ \text{compared to } \Delta T = 1 \]
E2 Gamma Transitions

Collectively enhanced E2

rotational band

vibrational band
E2 Transition Rates

E2 often collectively enhanced

$^{40}$Ca Inter-and Intraband E2 Transitions (W.u.)

Strong E2 transitions within bands
Weaker between different bands because of different intrinsic structure of band heads
some band mixing
SM: No $E$ transitions in odd-A (odd-$n$) nuclei, since $e(n)=0$

$E1 \rightarrow$ effective charges for $n$, $p$ due to c.m. motion

$$\hat{O}_{E1} = \sum_i \left( \frac{e}{2} \right) t_z \left( \vec{r}_i - \vec{R}_{cm} \right) \rightarrow e_{\text{eff}} \left( \begin{array}{c} p \\ n \end{array} \right) = \left( \begin{array}{c} +e/2 \\ -e/2 \end{array} \right)$$

$E2 \rightarrow$ core polarization $\rightarrow$ mixing of states

$$\psi \left( \begin{array}{c} 1^+ \\ 2 \\ 2 \end{array} \right) = \alpha \psi \left( ^{16}O_{gs} \right) \phi(2s_{1/2}) + \beta \psi \left( ^{16}O_{1^+} \right) \phi(1d_{5/2})$$

$$\psi \left( \begin{array}{c} 5^+ \\ 2 \\ 2 \end{array} \right) = \gamma \psi \left( ^{16}O_{gs} \right) \phi(1d_{5/2}) + \delta \psi \left( ^{16}O_{2^+} \right) \phi(2s_{1/2}) + \varepsilon \psi \left( ^{16}O_{2^+} \right) \phi(1d_{5/2})$$

$$\Gamma = (\gamma \beta + \alpha \delta) \Gamma \left( ^{16}O_{2^+} \rightarrow ^{16}O_{gs} \right) \quad \text{Here } 2^+ \text{ is collective}$$
Examples of Magnetic Transitions

\begin{align*}
J^\pi & \quad \text{keV} \\
\frac{1}{2}^- & \quad 315 \\
\frac{9}{2}^+ & \quad 0 \\
\frac{11}{2}^- & \quad 315 \\
\frac{3}{2}^+ & \quad 159 \\
\frac{1}{2}^+ & \quad 0
\end{align*}

- M4 (47%)
- β− (16%)
- E2
- M1
- M4
Gamma Angular Correlations
Gamma-Ray Angular Emission Patterns

Intensity $P(\ell)$ of multipole radiation ("# photons) is proportional to $\left| \text{solid sph. harmonics} \right|^2$

Same for $E\ell$, $M\ell$ ($P \propto E\times B$ Poynting)

Arbitrary quantization axis!

\begin{align*}
F_{m}^{\ell}(\theta) &= \left| X_{m}^{\ell}(\theta) \right|^2 = \\
&= \frac{1}{2\ell(\ell + 1)} \left[ \ell(\ell + 1) - m(m + 1) \right] \left| Y_{m+1}^{\ell} \right|^2 + \\
&\quad + \left[ \ell(\ell + 1) - m(m - 1) \right] \left| Y_{m-1}^{\ell} \right|^2 + 2m^2 \left| Y_{m}^{\ell} \right|^2 \\

F_{0}^{\ell} &= \frac{3}{8\pi} \sin^2 \theta \\
F_{1}^{\ell} &= \frac{3}{16\pi} \left( 1 + \cos^2 \theta \right) \\
F_{\pm 1}^{2} &= \frac{5}{16\pi} \left( 1 - 3\cos^2 \theta + 4\cos^4 \theta \right) \\
F_{\pm 2}^{2} &= \frac{5}{16\pi} \left( 1 - \cos^4 \theta \right) \\
\end{align*}
Example: 0-1-0 $\gamma$ Cascade

Probability for population of intermediate $m$ state by $\gamma_1$:

$$P\left(\frac{I_i}{m_i} \rightarrow \frac{I'}{m'}\right) = \sum_{m_i} \left(\frac{I_i}{m_i} \frac{\ell_1}{m_1} \frac{I'}{m'}\right)^2 F_{m_1}^{\ell_1}(\theta = 0)$$

$$W(\theta) = \sum_{m_i,m_f} P\left(\frac{I_i}{m_i} \rightarrow \frac{I'}{m'}\right)P\left(\frac{I'}{m'} \rightarrow \frac{I_f}{m_f}\right) \quad \text{no } m' \text{ state interference}$$

$$= \sum_{m_i,m_f} P\left(\frac{I_i}{m_i} \rightarrow \frac{I'}{m'}\right) \left(\frac{I'}{m'} \frac{\ell_2}{m_2} \frac{I_f}{m_f}\right)^2 F_{m_2}^{\ell_2}(\theta) =$$

$$= \sum_{m_i,m',m_f} \left(\frac{I'}{m'} \frac{\ell_2}{m_2} \frac{I_f}{m_f}\right)^2 \left(\frac{I_i}{m_i} \frac{\ell_1}{m_1} \frac{I'}{m'}\right)^2 F_{m_2}^{\ell_2}(\theta)F_{m_1}^{\ell_1}(0)$$

Cascade $I_i = I_f = 0, \ I' = 1$  
$\ell_1 = \ell_2 = 1 \Rightarrow m_i = m_f = 0$;  
$\begin{pmatrix} 0 & \ell_1 \\ m_1 & 1 \end{pmatrix} = \delta_{m_1 m'}$;  
$\begin{pmatrix} 1 & \ell_2 \\ m'_2 & 0 \end{pmatrix} = \delta_{m_2,-m'}$

$$W(\theta) = F_{\pm 1}^1(0) \left[ F_1^1(\theta) + F_{-1}^1(\theta) \right] \propto (1 + \cos^2 \theta)$$
Example: Rotational Cascade $4^+-2^+-0^+$

Calculate the angular gamma-gamma correlation function $W(\theta)$. Deduce the coefficients in the expansion

$$W(\theta) = \sum_{n=0}^{\infty} A_n \cdot P_n(\cos \theta)$$