SURVEY

$\gamma-\gamma$ Correlations in $e^+-e^-$ Annihilation and Nuclear Decay
**Positronium and e^+e^- Annihilation**

**Para Positronium**

\[ \vec{I} = 0 \]

\[ \vec{s}_2 \]

\[ E = 0 \text{ eV} \]

**Ortho Positronium**

\[ \vec{I} = 1 \]

\[ \vec{s}_2 \]

\[ E = 8 \cdot 10^{-4} \text{ eV} \]

\[ \sigma_{2\gamma} = \pi r_0^2 \cdot \frac{\nu_{e^+e^-}}{c} \]

\[ \sigma_{2\gamma}/\sigma_{3\gamma} = 372 \]

\[ \tau_{3\gamma}(n) = 1.4 \cdot 10^{-7} n^3 \text{ sec} \]

\[ r_0 = 2.818 \text{ fm, class. el. radius} \]

**Decay at rest:**

\[ 2E_\gamma \approx E_0 \approx 1.022 \text{ MeV} \]

\[ \theta_{12} \approx 180^0 \]

2-body decay → line spectrum in \( E_\gamma \) and \( \theta_{12} \)

\[ \tau_{2\gamma}(n) = 1.25 \cdot 10^{-10} n^3 \text{ sec} \]

\( n = \text{principal quantum } \# \)

3-body decay → continuum in \( E_\gamma \) and \( \theta_{12} \)
Radiation Detectors for Medical Imaging

Positron emission tomographic (PET) virtual slice through patient’s brain

Administer to patient radioactive water: $H_2^{17}O$ radioactive acetate: $^{11}CH_3COOX$

Observe $^{17}O$ or $^{11}C$ $\beta^+$ decay by

$$e^+ + e^- \rightarrow 2\gamma(511\, keV)$$

Positron $e^+$ (anti-matter) annihilates with electron $e^-$ (its matter equivalent of the same mass) to produce pure energy (photons, $\gamma$-rays). Energy and momentum balance require back-to-back ($180^0$) emission of 2 $\gamma$-rays of equal energy

$\gamma$ Detectors (NaI(Tl))
Top left: PET imaging experiment setup with two 1.5”x1.5” NaI(Tl) detectors (BICRON) on a slotted correlation table. A “point-like” $^{22}\text{Na} \gamma$ source can be hidden from view.
Top right: $^{22}\text{Na} \gamma$ spectrum measured with NaI$_1$.
Bottom right: $\gamma - \gamma$ angular correlation measurement.

Second correlation setup: NaI(Tl) vs. HPGe
$E_1 - E_2 - \Delta t$ Multi-Parameter Measurement

$\gamma - \gamma$ cascade ($^{60}$Ni) or PET experiment, full 3D energy-time distribution.
$E_1 - E_2$ Dual-Parameter Measurement

$\gamma - \gamma$ cascade ($^{60}$Ni) or PET experiment, full 2D spectrum

Diagram showing the flow of signals through preamps, amplifiers, coincidence generators, and energy detectors, leading to data acquisition (DAQ).
2D Parameter Measurement

2D Scatter Plot: Each point represents one event \{E_1, E_2\}

Projection on \( E_1 \) axis

Projection on \( E_2 \) axis
Gated E₁-E₂ Coincidence Measurement

$\gamma-\gamma$ cascade ($^{60}$Ni) or PET experiment, gates on $\gamma_1$ and $\gamma_2$ lines

Diagram:
- Amp
- SCA
- Gate Generator
- Coincidence
- OR
- Energy 1
- Energy 2
- Strobe
- Pulser
- Counter
- Pulse Generator
- Dead time measurement
- DAQ
Absolute Activity Measurement

Activity $A$ [disintegrations/time], independent radiation types $i = 1, 2$ detection probabilities $P_i$

- $N_1 = A \cdot P_1$
- $N_2 = A \cdot P_2$
- $P_{12} = P_1 \cdot P_2$
- $N_{12} = A \cdot P_{12}$

Individual rates

Coincidence rate

Singles / Coincidence

$$A = \frac{N_1 \cdot N_2}{N_{12}} = \frac{A \cdot P_1 \cdot A \cdot P_2}{A \cdot P_{12}}$$
SURVEY

\(\gamma - \gamma\) Correlations in Nuclear Decay
Symmetries of the Nuclear Mean Field

Nucleon-nucleon forces are dominantly radial

\[ U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|) \]

**Nuclear mean field per nucleon**

\[ \tilde{U} = \frac{1}{2A} \sum_{i,j} U(|\vec{r}_i - \vec{r}_j|) \]

\[ U(\vec{r}) \approx \tilde{U}(r) \text{ central potential, no angle dependence} \]

→ **Invariance against space inversion, rotations**
→ **Conservation of parity and angular momentum**

\[ (= \text{quantum duality}) \]

\[ \hat{H} = \sum_{i=1}^{A} \left\{ \frac{\hat{p}_i^2}{2m_i} + \tilde{U}(|\vec{r}_i|) \right\} \rightarrow [\hat{H}, \hat{\Pi}] = 0 = [\hat{H}, \hat{R}] \]

because \[ \hat{\Pi} \hat{p}_{ix}^2 = -\hbar^2 \frac{\partial^2}{\partial(-x_i)^2} = \hat{p}_{ix}^2 \] and \[ \tilde{U}(|\vec{r}_i|) = \tilde{U}(|\vec{r}_i|) \]

and \[ \hat{R}_z \tilde{U}(|\vec{r}_i|) = \tilde{U}(|\vec{r}_i|) = \tilde{U}(|\vec{r}_i|) \]

and \[ \hat{R}_z \hat{p}_i^2 = \hat{p}_i^2 \]
Stationary Nuclear States

Consequences of space-inversion and rotational invariance:
Stationary states $\hat{H}$ eigen states are also eigen states of $\hat{\Pi}$ and $\hat{L}, \hat{L}_z$

Radial $(r)$ and angular $(\theta, \phi)$ degrees of freedom are independent $\Rightarrow$
Separate d.o.f in Schrödinger Equ. $\Rightarrow$ Product wave functions

$\hat{H} j_n(r) = E_n j_n(r)$ stationary states

Operator for rotations: $\hat{R}_z(\varphi_z) = e^{-i \varphi_z \hat{L}_z}$

$\left[ \hat{H}, \hat{R}_z(\varphi_z) \right] = 0 \Rightarrow \left[ \hat{H}, \hat{L}_z \right] = 0 = \left[ \hat{H}, \hat{L}^2 \right]$

$\hat{L}^2 Y_m^\ell(\theta, \phi) = \ell(\ell + 1)\hbar^2 \cdot Y_m^\ell(\theta, \phi)$

$\hat{L}_z Y_m^\ell(\theta, \phi) = m\hbar \cdot Y_m^\ell(\theta, \phi)$

$\hat{\Pi} Y_m^\ell = (-)^\ell Y_m^\ell$ Parity $\pi = (-)^\ell = \begin{cases} + \\ - \end{cases}$

$\psi_{n, \ell, m}(r, \theta, \phi) = j_n^\ell(r) \cdot Y_m^\ell(\theta, \phi)$

Spherical Harmonics or (axial symmetry) Legendre Polynomials
Electromagnetic Nuclear Transitions

Conserved: Total energy \( (E) \), total angular momentum \( (I) \) and total parity \( (\pi) \):

\[
E_i = E_f + E_\gamma \quad I_i = I_f + I_\gamma \quad \pi_i = \pi_f \cdot \pi_\gamma
\]

\[
\begin{align*}
E_i & = 2^+ \\
1^- & = 0^+ \\
E_f & = 1^- \\
0^+ & = 0^+ \\
\end{align*}
\]

Initial Nucl. State \( j_{22}(r) \cdot Y_{m_j}^{2}(\theta, \varphi) \)

Final Nucl. State \( j_{11}(r) \cdot Y_{m_f}^{1}(\theta, \varphi) \)

Photon WF \( \psi_{\Delta E}(r_\gamma) \cdot Y_{\mu=\Delta m}^{\ell=\Delta I}(\theta, \varphi) \)

Consider here only \textit{electric} multipole transitions. Neglect weaker \textit{magnetic} transitions due to changes in current distributions.
Protons in nuclei = moving charges $\rightarrow$ emits electromagnetic radiation

Propagating Electric Dipole Field

E. Segré: *Nuclei and Particles*, Benjamin&Cummins, 2nd ed. 1977
Selection Rules for Electromagnetic Transitions

Conserved: Total energy ($E$), total angular momentum ($I$) and total parity ($\pi$):

$$E_i = E_f + E_\gamma \quad \vec{I}_i = \vec{I}_f + \vec{I}_\gamma \quad \pi_i = \pi_f \cdot \pi_\gamma$$

Quantization axis: $z$ direction. Physical alignment of $I$ possible (B field, angular correlation)

Coupling of Angular Momenta

Angular momenta $\vec{I}_i, \vec{I}_f$

direction undetermined.

Projections conserved

$m_i, m_f, \quad m_\gamma = m_i - m_f$

$$\rightarrow |\vec{I}_i - \vec{I}_f| \leq \ell_\gamma \leq |\vec{I}_i + \vec{I}_f|$$

Electric dipole radiation [$\ell_\gamma = 1(\hbar)$]:

$I_f = I_i, \quad I_i \pm 1(\hbar) \quad m_f = m_i, m_i \pm 1$
Spherical Harmonics $P_1^{|m|}\cos (m\phi)$

Plot $\text{Re } Y^m_m \propto P_1^{|m|}\cos (m\phi)$

$$\vec{r}(r, \theta, \phi) = \{1, P_1^{|m|}(\theta), \cos(m\phi)\}$$

$x = r \sin(\theta) \cos(\phi)$

$y = r \sin(\theta) \sin(\phi)$

$z = r \cos(\theta)$

$$|Y^1_{\pm 1}(\theta)|^2 \propto \frac{1}{2} \left(1 + \cos^2 \theta\right)$$ Emission in $z$ direction

$$|Y^1_0(\theta)|^2 \propto \sin^2 \theta$$ Emission perpend. to $z$

$$W(\theta) = \frac{P(m = 0)}{1/3} |Y^1_0(\theta)|^2 + \frac{P(m = \pm 1)}{2 \times 1/3} |Y^1_1(\theta)|^2$$

Gamma-Gamma Correlations
γ–γ Angular Correlations

Simple example: γ cascade 0 → 1 → 0
Mostly Δm=±1 emitted in z-direction

General Expression for γ–γ angular correlation

\[ W_{\gamma_1\gamma_2}(\theta) = \sum_{n=0}^{\ell} A_{2n} P_{2n}(\cos \theta) \propto 1 + A'_2 \cos^2 \theta + \ldots \]

Typically: n ≤ 2. Legendre Polynomials \( P_n(\cos \theta) \)
E2 $\gamma-\gamma$ Angular Correlations

Example: Rotational E2-$\gamma$ cascade, $\Delta m=\pm 2$ maximally emitted in z-direction

$\ell = 2 \rightarrow$ highest order $P_4$

$W(\theta) = 1 + 0.1020 \cdot P_2(\cos \theta) + 0.0091 \cdot P_4(\cos \theta)$

Anisotropy

$A_{\gamma\gamma} := \frac{W(90^0) - W(180^0)}{W(180^0)} = \sum_{n=1}^{n_{\text{max}}} A'_{2n}$

No More Correlations
Gamma-Gamma Correlations