Spectral Analysis
Basic Statistics

Reading Assignment : Knoll, Ch. 3; Bevington, Chs.1-3

Data Reduction ("Analysis")

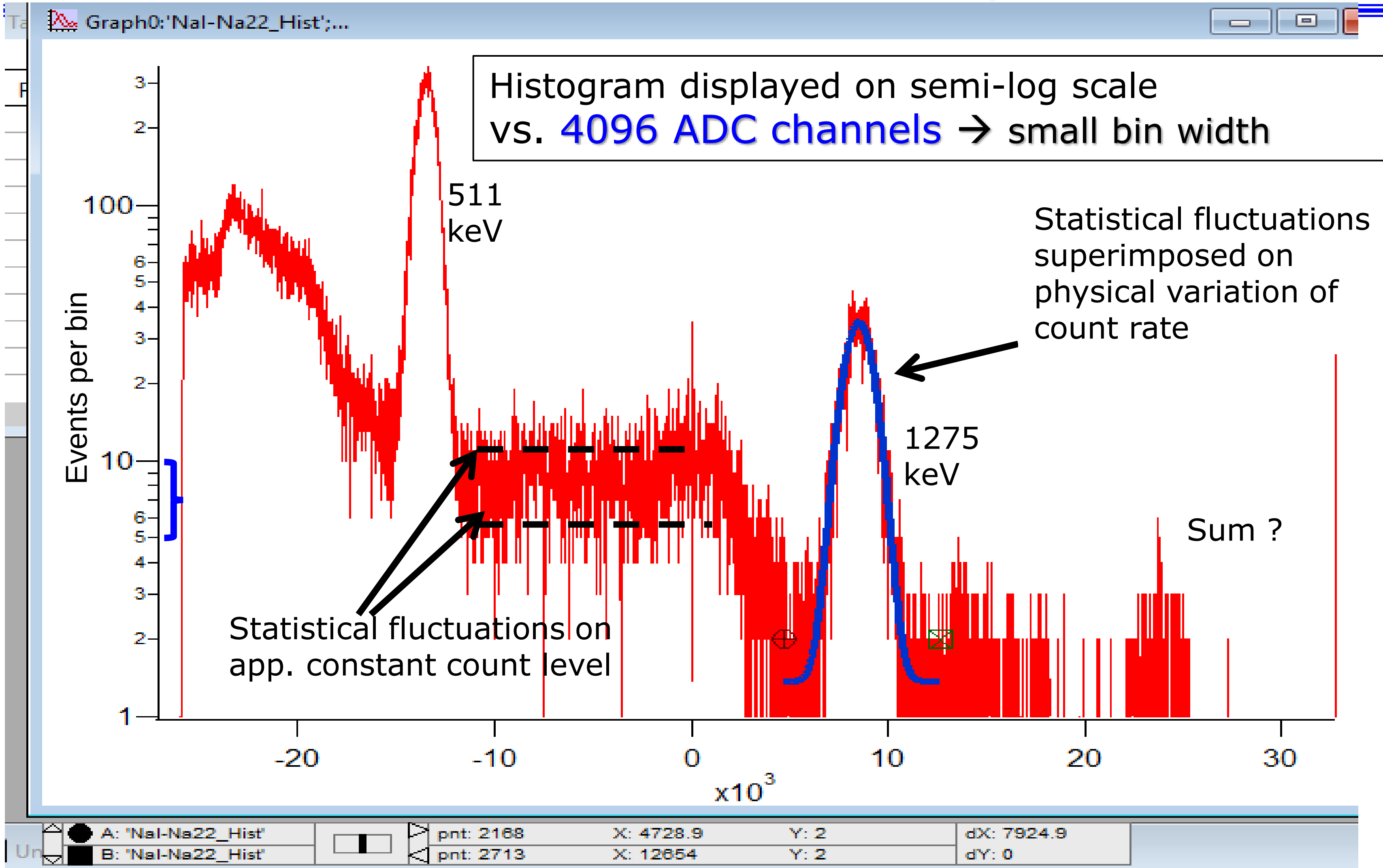
Acceptable data analysis includes evaluation and discussion of uncertainties of parameters providing a "reduced" description of measured data.

Example:
Sample mean (expectation) value, variance (standard deviation)

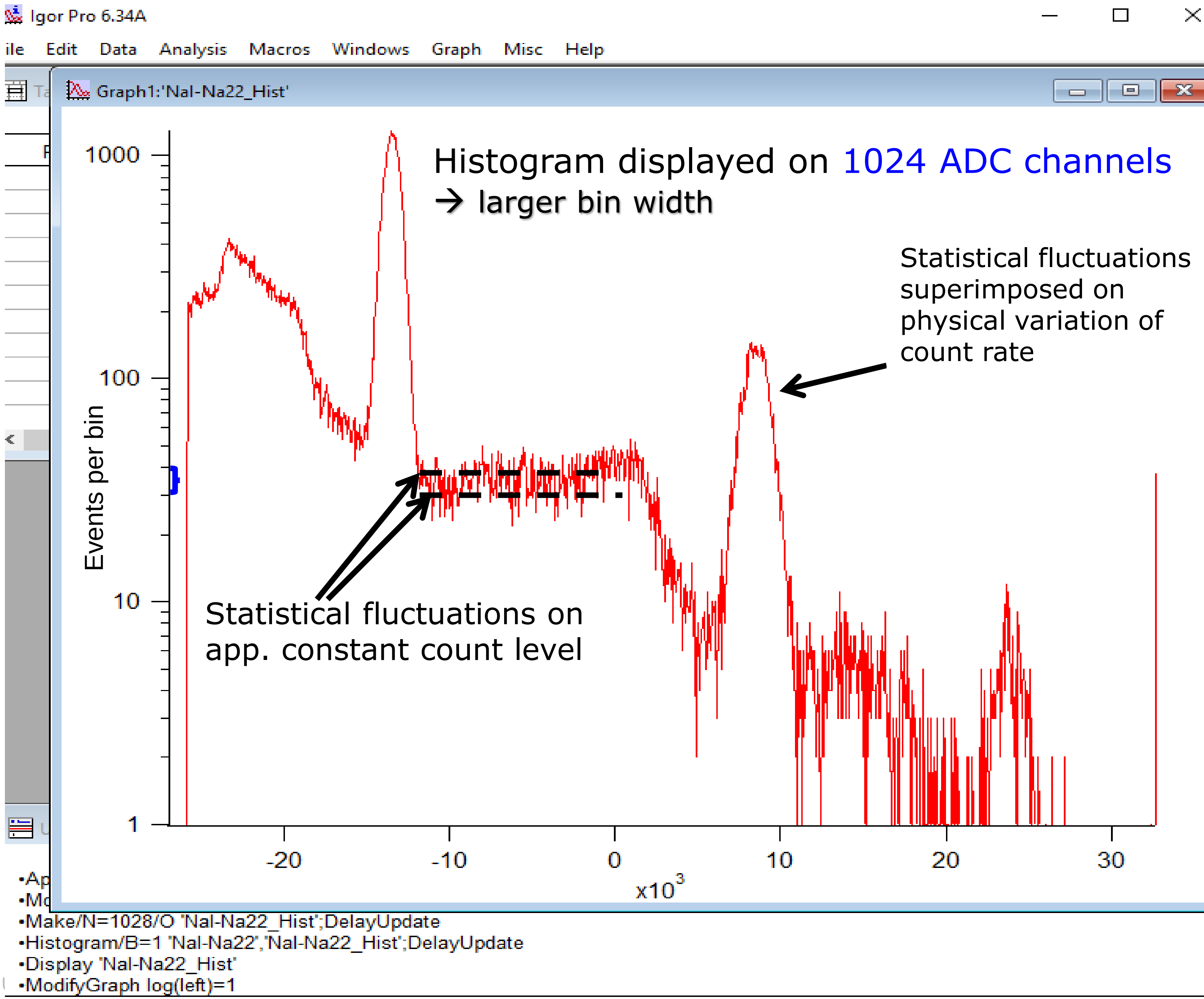
- Statistical (random) uncertainties → Precision

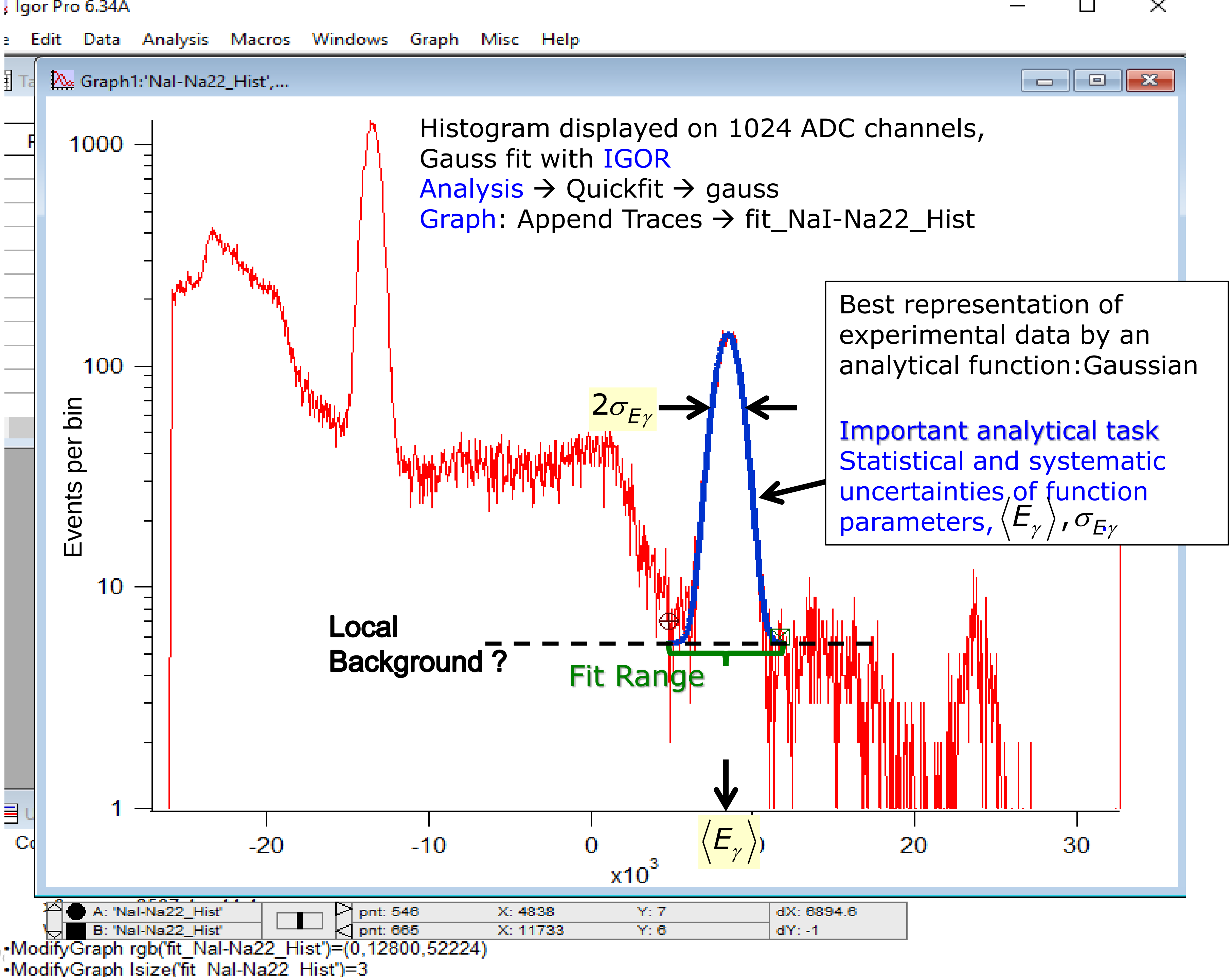
- Systematic (instrument-inherent or analytical) → Accuracy
  biased detector response, bad resolution,
  poor analysis procedures (bad fit function, poor choice of fit range, wrong Bckg. function, ..)

First ANSEL Data: Na-22 γ Spectrum



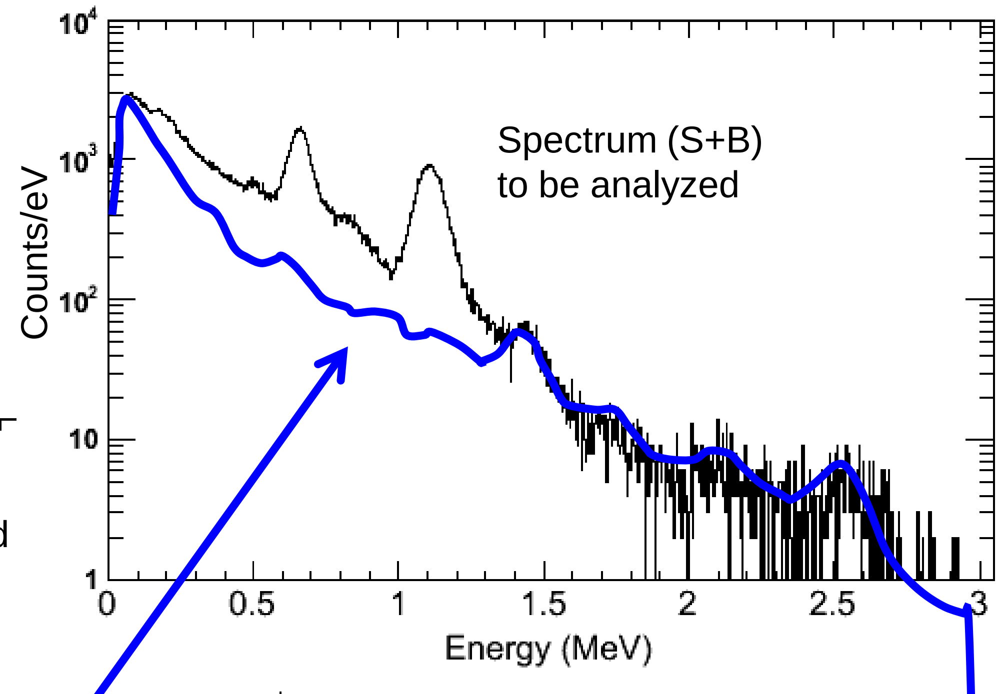
Coefficient values ± one standard deviation
y0 =1.3643 ± 0.185
A =33.626 ± 0.393
x0 =8528.8 ± 10
width =1113.3 ± 16.6
AppendToGraph 'fit_NaI-Na22_Hist' vs 'fit_NaI-Na22_Hist'
ModifyGraph rgb('fit_NaI-Na22_Hist')=(0,12800,52224)

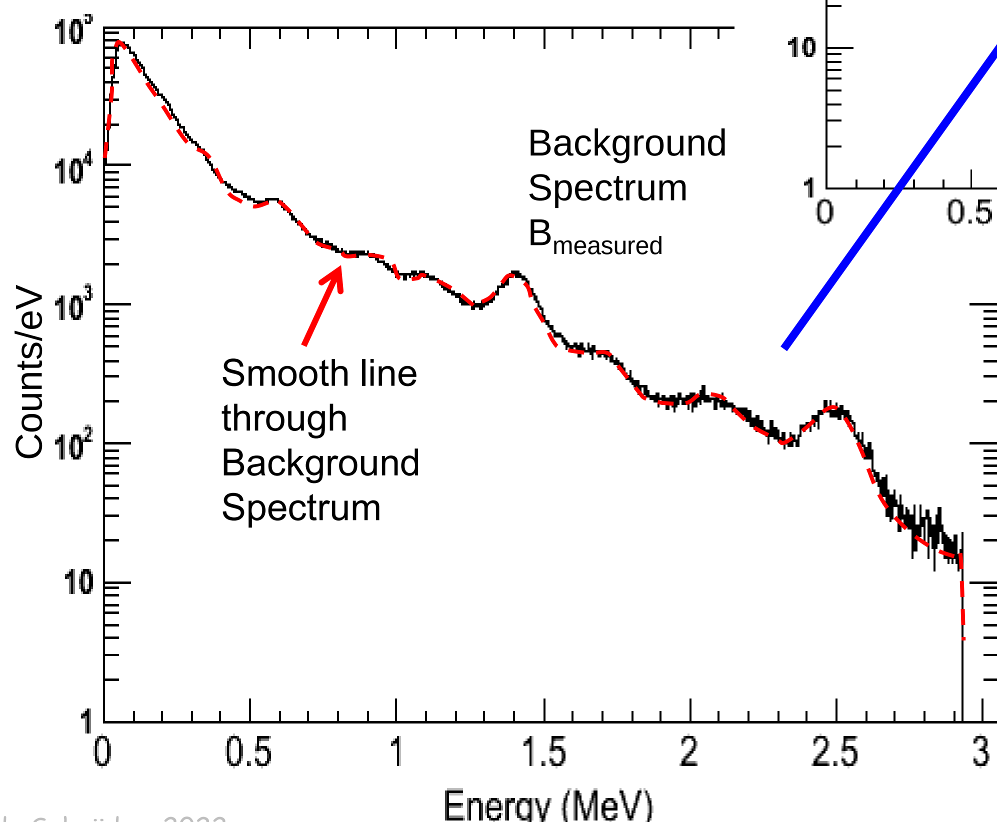

Igor Pro 6.34A
ile Edit Data Analysis Macros Windows Graph Misc Help
Graph1:'NaI-Na22_Hist'
Histogram displayed on 1024 ADC channels
→ larger bin width
Statistical fluctuations superimposed on physical variation of count rate
Statistical fluctuations on app. constant count level
Events per bin
1000
100
10
1
-20
-10
0
10
20
30
x10^3
•Make/N=1028/O 'NaI-Na22_Hist';DelayUpdate
•Histogram/B=1 'NaI-Na22','NaI-Na22_Hist';DelayUpdate
•Display 'NaI-Na22_Hist'
•ModifyGraph log(left)=1


Igor Pro 6.34A
Edit Data Analysis Macros Windows Graph Misc Help
Graph1:'NaI-Na22_Hist',...
Histogram displayed on 1024 ADC channels,
Gauss fit with IGOR
Analysis → Quickfit → gauss
Graph: Append Traces → fit_NaI-Na22_Hist
Best representation of experimental data by an analytical function: Gaussian
Important analytical task
Statistical and systematic uncertainties of function parameters, $\langle E_\gamma \rangle, \sigma_{E\gamma}$
$2\sigma_{E\gamma}$
Events per bin
1000
100
10
1
Local Background ?
Fit Range
-20
-10
0
20
30
$\langle E_\gamma \rangle$
x10³
A: 'NaI-Na22_Hist'
B: 'NaI-Na22_Hist'
pnt: 546
pnt: 665
X: 4838
X: 11733
Y: 7
Y: 6
dX: 6894.6
dY: -1
•ModifyGraph rgb('fit_NaI-Na22_Hist')=(0,12800,52224)
•ModifyGraph lsize('fit_NaI-Na22_Hist')=3

Signal-Plus-Background Spectrum

Measurement in radiation environment → always superpositions of components from various sources → Recognize patterns, measure Background *in situ*





Produce a difference spectrum $(S+B)-\alpha\cdot B_{measured}$
Scaling factor $\alpha(E)$?
from corrected ratio of run durations, or from local fit.

Radioactive Decay as Poisson Process

Useful when only a mean count rate is known: decay, background counts, or reaction.

^{137}Cs = unstable isotope, decays with
$t_{1/2}$ = 27 years → **p** = ln2/27 = 0.026/a = **$8.2 \cdot 10^{-10} s^{-1}$** → small

Sample of 1 μg: N = 10^{15} nuclei (=trials for decay)
How many will decay?

$$\mu = N \cdot p = \mathbf{8.2 \cdot 10^{+5}\ s^{-1}}$$

Count rate estimate dN/dt = (**$8.2 \cdot 10^{+5}$ ± 905**) **s^{-1}**

estimated

Probability for m decays $P(\mu, m)$ =

$$P_{Poisson}(\mu, m) = \frac{\mu^m \cdot e^{-\mu}}{m!} = \frac{(8.52 \cdot 10^5)^m \cdot e^{-8.52 \cdot 10^5}}{m!}$$

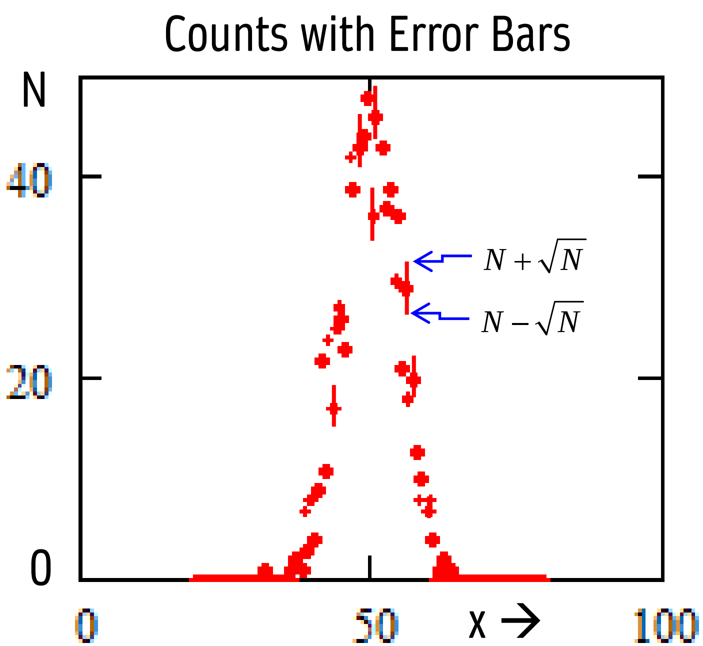
Poisson Probability Distribution

Limit of binomial distribution $Lim_{p\to 0, N\to\infty} P_{binomial}(N,m) = P_{Poisson}(\mu, m)$

Probability for observing m events when average is <m> = μ

$$P_{Poisson}(\mu, m) = \frac{\mu^m \cdot e^{-\mu}}{m!}$$

μ = <m> = N·p for $N \to \infty$
and $\sigma^2 = \mu$



For radioactive decays $[\Delta t^{-1}] \to p = \frac{A}{N}$

$p \ll 1 \to \sigma_m^2 \approx \langle m \rangle \ \#counts$

Observe N counts (events) →
→ statistical uncertainty is $\pm\sigma_N = \pm\sqrt{N}$

Functions of Stochastic Variables

Random independent variables $N_1, N_2,....,N_n$
corresponding variances $\sigma_1^2, \sigma_2^2,....,\sigma_n^2$
Function $f(N_1, N_2,....,N_n)$ of random variables: Uncertainty $\Delta N_i \rightarrow \Delta f(\{N_i\})$

Gauss' law of error propagation:

$$\sigma_f \approx \left\{ \underbrace{\left(\frac{\partial f}{\partial N_1}\right)^2 \sigma_1^2}_{\left(\Delta f\,|_{N_2,N_3,..}\right)^2} + \underbrace{\left(\frac{\partial f}{\partial N_2}\right)^2 \sigma_2^2}_{\left(\Delta f\,|_{N_1,N_3,..}\right)^2} + ... + \underbrace{\left(\frac{\partial f}{\partial N_n}\right)^2 \sigma_n^2}_{\left(\Delta f\,|_{N_1,N_2,..,N_{n-1}}\right)^2} \right\}^{1/2}$$

$$\left(\Delta f\,|_{N_2,N_3,..}\right)^2 + \left(\Delta f\,|_{N_1,N_3,..}\right)^2 + .. + \left(\Delta f\,|_{N_1,N_2,..,N_{n-1}}\right)^2$$

Further terms if N_i are not independent (→ correlations, co-variance tensor)
Otherwise, individual independent component variances $(\Delta f)^2$ add.

Experimental Mean Count Rate and Variance

What can be measured: ensemble (sampling) averages (expectation values) and uncertainties

Task: ^{236}U (0.25mg) source, count # α particles emitted during N = 10 time intervals Δt (samples @ Δt ≈ 1 min). λ = ??

n	n-<n>	$(n-\langle n\rangle)^2$
36076	129.6	16796.16
35753	-193.4	37403.56
35907	-39.4	1552.36
36116	169.6	28764.16
35884	-62.4	3893.76
36136	189.6	35948.16
35741	-205.4	42189.16
35640	-306.4	93880.96
36124	177.6	31541.76
36087	140.6	19768.36
35946	-1.5E-12	**3463.76**
<n>	**<n-<n>>**	**$\sigma^2{}_n$**

Average count n in a sample of a population:

$$\langle n\rangle = \frac{1}{N}\cdot\sum_{i=1}^{N} n_i \quad (\neq \langle n\rangle_{population}\ unknown)$$

Variance of n in each of the M individual samples

$$s^2 = \sigma^2 = \frac{1}{N-1}\cdot\sum_{i=1}^{N}(n_i - \langle n\rangle)^2 \quad \rightarrow (N = N_m,\ m = 1,\ldots,M)$$

Variance ("error") of the sample average $\langle n\rangle \neq \langle n\rangle_{population}$

$$\sigma_n^2 = \frac{s^2}{M} = \frac{1}{M(N-1)}\cdot\sum_{i=1}^{N}(n_i - \langle n\rangle)^2$$

$$Std.deviation: \sigma_n = \sqrt{\sigma_n^2} \approx \sqrt{<n>/N} = 59$$

$$Result: \langle n\rangle \approx \langle n\rangle_{pop} = (35946 \pm 59)\,\mathrm{min}^{-1}$$

"Error" of <n> much smaller than σ^2. It is reduced by 1/10 for 100 times larger sample

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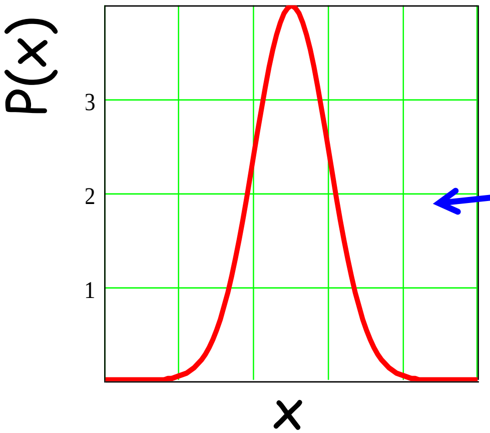
Sample Statistics (Simulation)

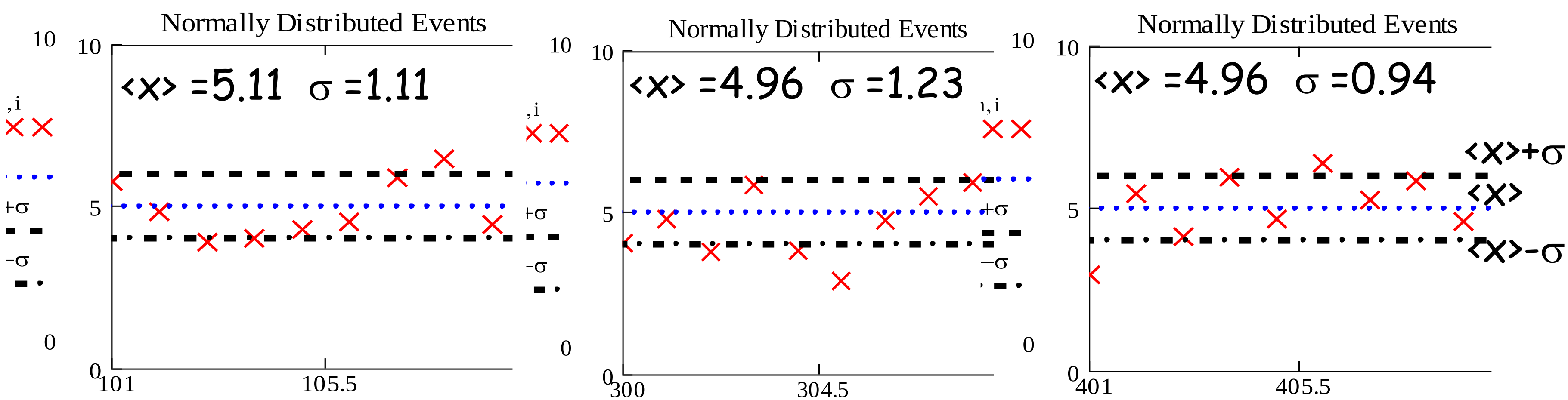


Assume true population distribution for variable x

$$P(x)=\frac{1}{\sqrt{2\pi v_X^2}}\cdot\exp\left\{-\frac{\left(x-\langle x\rangle_{pop}\right)^2}{2v_X^2}\right\}$$

with true ("population") mean $\langle x\rangle_{pop} = 5.0$, $\nu_x = 1.0$

Sample of M=three 10-count "measurements" of some variable x:



Equally weighted sample average $\langle x\rangle = (5.11+4.96+4.96)/3 = 5.01$

Sample variance $s^2=\sigma^2 = [(5.11-5.01)^2+\mathbf{2}(4.96-5.01)^2]/2 = 0.01$ **s = 0.0075**

$\sigma_x^2 = 0.0075/3 = 0.0025$ $\sigma_x = 0.05$ → Result: $\langle x\rangle_{pop} \approx 5.01 \pm 0.05$

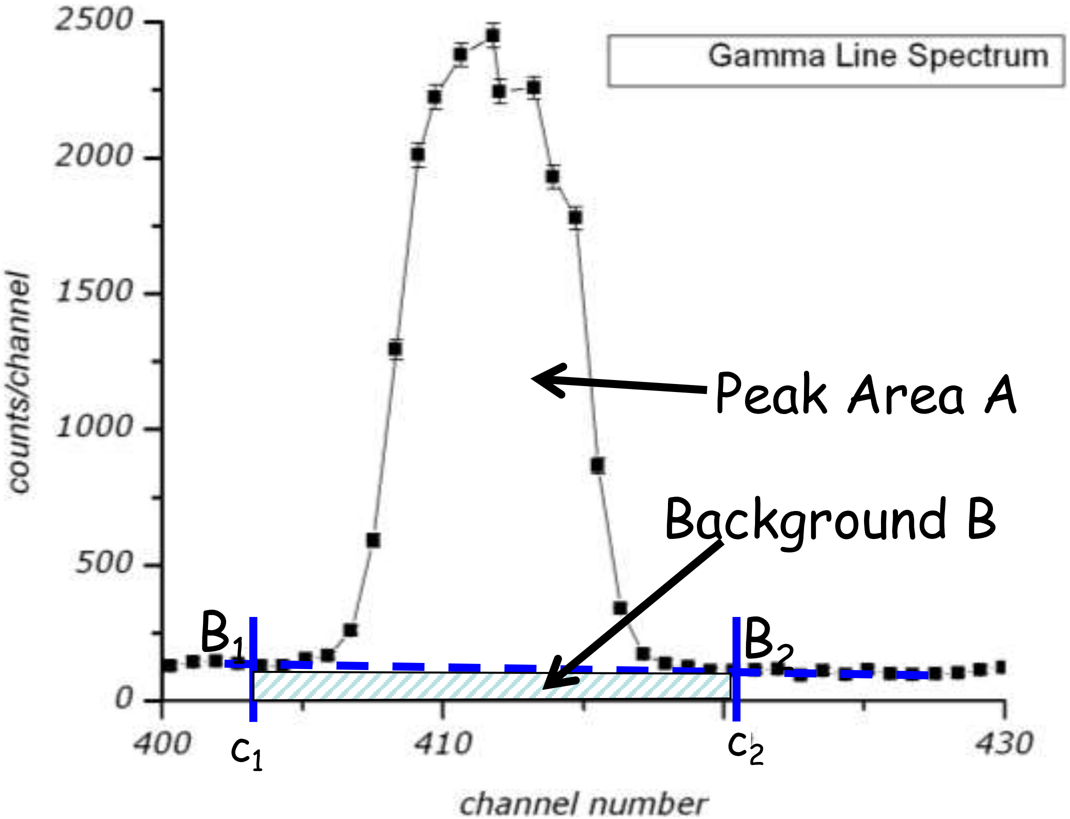
Could replace with one measurement 30 counts long.

Example: Spectral Analysis (Local Bck Subtr.)

Add or subtract 2 Poisson-distributed numbers N_1 and N_2:
Variances σ^2 always add

$$N := \left[N_1 \pm \sqrt{N_1}\right] \pm \left[N_2 \pm \sqrt{N_2}\right] \triangleq \left(N_1 \pm N_2\right) \pm \sqrt{N_1 + N_2}$$

Std. dev σ_1 Std. dev σ_2 Std. dev $\sigma_{1\pm2}$



Analyze peak in range channels $c_1 - c_2$: beginning of background left and right of peak

$$n = c_1 - c_2 + 1.$$

Total area c_1 - c_2 → N_{12}

$N(c_1)=B_1$, $N(c_2)=B_2$,

Linear (≈constant) background

$B = n(B_1+B_2)/2$

$$\text{Peak area } A = \sum counts_i / ch$$

$$A = N_{12} - n \cdot \left(B_1 + B_2\right) / 2$$

Stat uncertainty std.dev.

$$\sigma_A = \sqrt{N_{12} + n \cdot \left(B_1 + B_2\right) / 2}$$

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Stochastic Observables

2 sources of stochastic observables *x* in nuclear science:

1) Nuclear phenomena are governed by quantal wave functions and inherent statistics
2) Detection of process occurs with imperfect efficiency ($\varepsilon < 1$) and finite resolution distributing sharp events x_0 over a range in x.

Stochastic observables *x* have a range of values with frequencies determined by (normalized) probability distribution P(x)

Characterize *P* by set of **moments** of *P*

$$\langle x^n \rangle = \int x^n \cdot P(x)dx; \quad n = 0, 1, 2, \ldots$$

with the **normalization** $\langle x^0 \rangle = 1$. First moment of P:

$$E(x) = \langle x \rangle = \int x \cdot P(x)\, dx$$

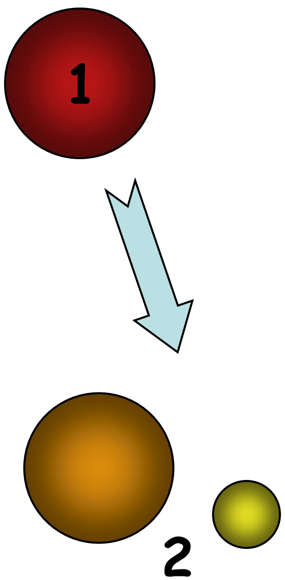
second **central moment** = "variance" of P(x): $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$

Uncertainty and Statistics

Nucleus is a quantal system described by a wave function $\psi(x,\ldots;t)$

$(x,\ldots;t)$ are the degrees of freedom of the system and time.

Probability density $dP(x,t)/dx$ (e.g., for x, integrate over others)



$$\frac{dP(x,t)}{dx} = |\psi(x,t)|^2$$

Normalization

$$P(\not{x},t) = \int_{-\infty}^{+\infty} dx \frac{dP(x,t)}{dx} = \int_{-\infty}^{+\infty} dx |\psi(x,t)|^2 = 1$$

$$\textit{Transition between states } 1 \rightarrow 2,\ \Gamma \approx \frac{\hbar}{2\pi} |\langle M_{12} \rangle|^2 \rho(E)$$

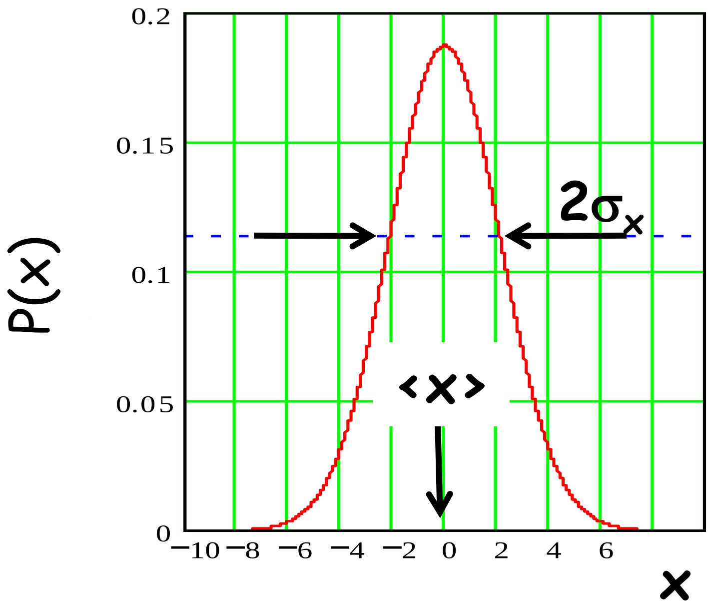
$$1 \textit{ is not a stationary state} \rightarrow \textit{finite width } \Delta E \sim \Gamma$$

$$\frac{dP_1(x,t)}{dx} = |\psi(x)|^2\, e^{-\frac{\Gamma}{\hbar} \cdot t} \propto e^{-\lambda \cdot t} \quad \textit{state 1 disappears}$$

$$\lambda = 1/\tau \quad \textit{mean lifetime } \tau$$

For different nuclei or different states of one nucleus, the probability rate λ for disappearance (decay rate) can vary over many orders of magnitude → no certainty for any single entity.

Normal Distribution of a Random Variable

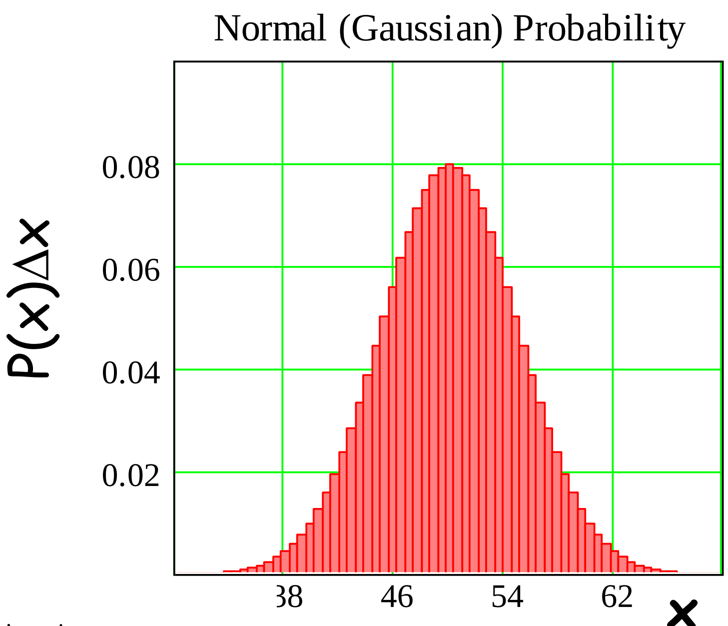


Continuous function or discrete distribution (over bins Δx=const)

$$P(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \exp\left\{ -\frac{\left(x - \langle x \rangle\right)^2}{2\sigma_X^2} \right\}$$

$$\Gamma_{FWHM} = 2\sigma_x \cdot \sqrt{2\ln 2} = 2.35 \cdot \sigma_x$$

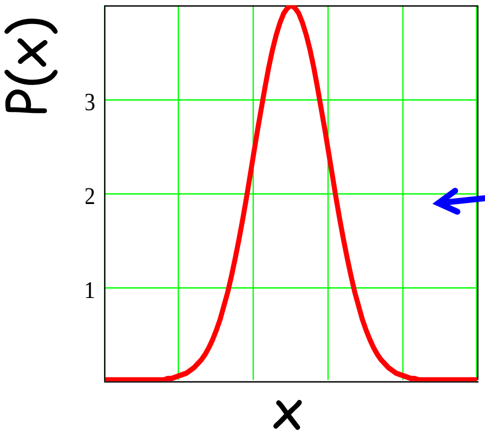
$$\sigma_x \; is \; NOT = uncertainty \; of \; \langle x \rangle!$$



Normalized (cumulative) probability

$$P(x < x_1) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \int_{-\infty}^{x_1} dx \exp\left\{ -\frac{\left(x - \langle x \rangle\right)^2}{2\sigma_X^2} \right\}$$

Sample Statistics (Simulation)

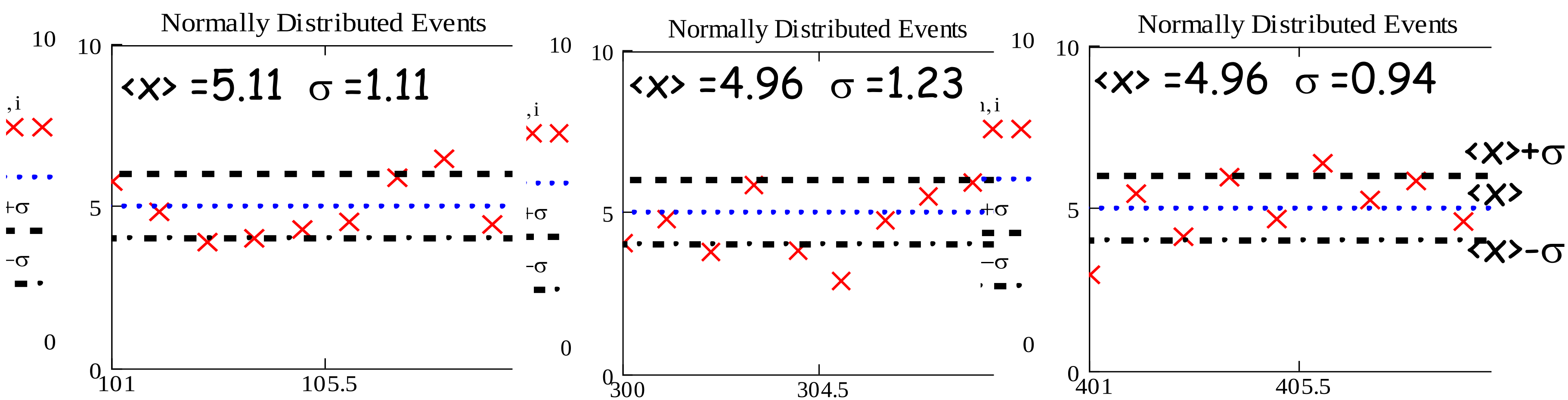


Assume true population distribution for variable x

$$P(x)=\frac{1}{\sqrt{2\pi v_X^2}}\cdot\exp\left\{-\frac{\left(x-\langle x\rangle_{pop}\right)^2}{2v_X^2}\right\}$$

with true ("population") mean $\langle x\rangle_{pop}$ = 5.0, ν_x =1.0

Sample of M=three 10-count "measurements" of some variable x:



Equally weighted sample average $\langle x\rangle$ = (5.11+4.96+4.96)/3 = 5.01

Sample variance $s^2=\sigma^2$ = [(5.11-5.01)²+**2**(4.96-5.01)²]/2 = 0.01 **s = 0.0075**

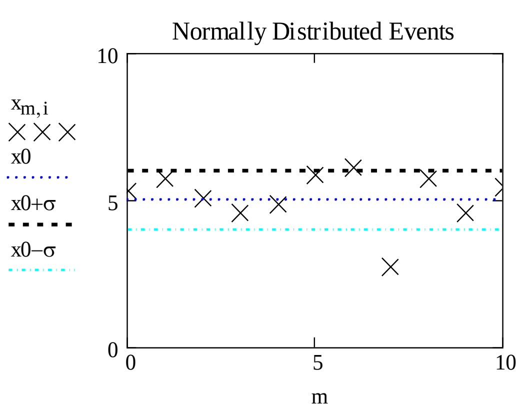
σ_x^2 = 0.0075/3 = 0.0025 σ_x = 0.05 → Result: $\langle x\rangle_{pop}$ ≈ 5.01 ± 0.05

Could replace with one measurement 30 counts long.

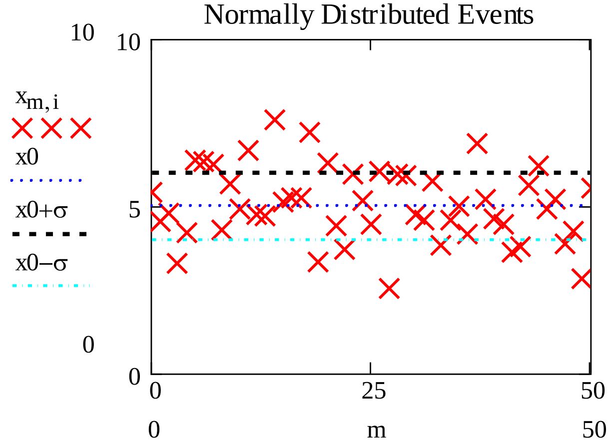
Example 2

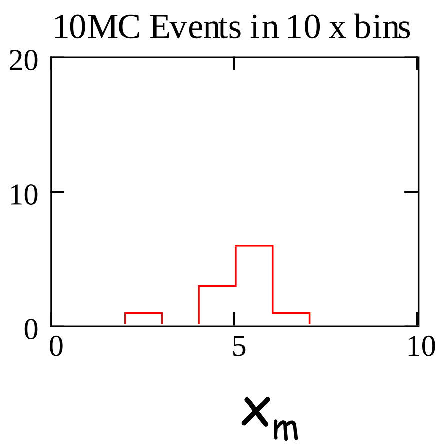
Sample size

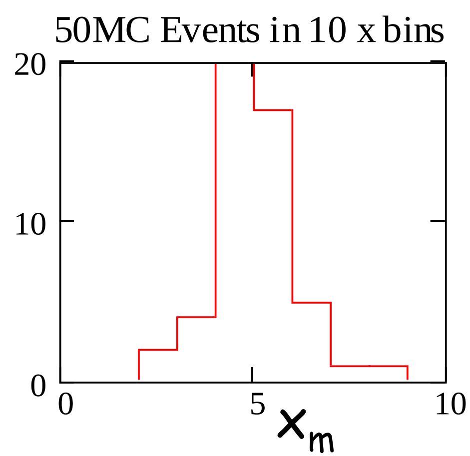
n = 10



n = 50







The larger the sample, the narrower the distribution of x values, the more it approaches the true Gaussian (normal) distribution.

Central-Limit Theorem

The means (averages) of different samples in the previous examples cluster together closely. → *general property of samples of stochastic variables:*

> The distribution of the **sample means** approaches a Gaussian normal distribution, if the size n of the sample increases, *regardless of the form of the original (population) distribution.*

The mean (average) of a distribution of stochastic data does not contain information on the actual shape of the distribution.

The average of any truly random sample of a population is already close to the true population average. Considering many samples, or large samples, narrows the choices. The Gaussian width becomes narrower for larger samples. → The standard error of the mean decreases as the sample size increases.

Probability theory →

Binomial Distribution

Integer random variable *m* = number of events, out of *N* possible.
Example: decay of *m* (from a sample of *N*) radioactive nuclei, or detection of *m* (out of *N*) photons arriving at detector.
Let
p = probability for a (one) success (decay of 1 nucleus, detection of 1 photon)

Choose an arbitrary sample of *m* trials out of *N* total trials (possibilities)

p^m = **probability for at** least *m* successes (observations)
$(1-p)^{N-m}$ = **probability for N-m failures** (survivals, not detected,…)

Probability for exactly *m* successes out of a total of *N* trials

$$P(m) \propto p^m \cdot (1-p)^{N-m}$$

How many ways can *m* events be 'chosen' out of N ? → Binomial coefficient

$$\binom{N}{m} = \frac{N!}{m!(N-m)!} = \frac{(N-m+1)\cdots N}{1\cdots m}$$

Total probability (expected success rate) for any sample of *m* identical events:

$$P_{binomial}(m) = \binom{N}{m} \cdot p^m \cdot (1-p)^{N-m}$$

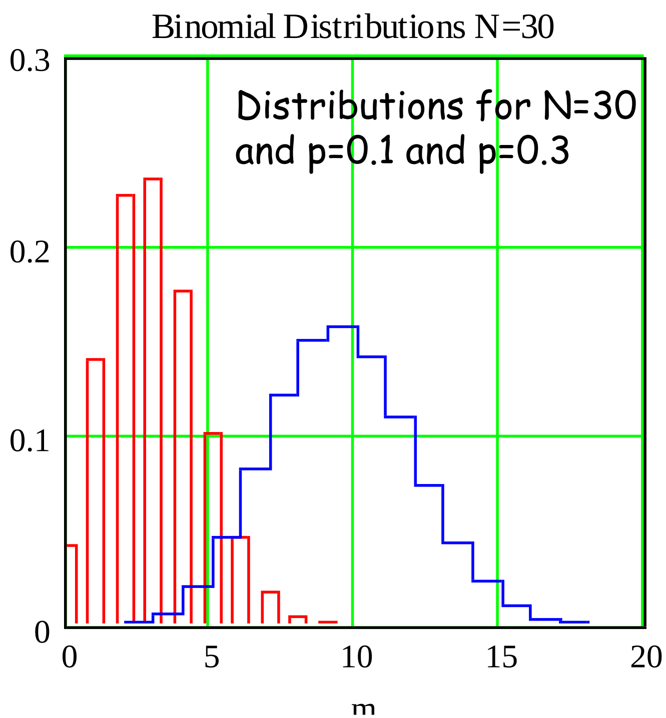
Distribution Moments and Limits

$$P_{binomial}(N,m,p) = \binom{N}{m} p^m (1-p)^{N-m}$$

Probability for m "successes" out of N trials, individual probability p

Normalization

$$1 = \sum_{m=0}^{N} P_{bin}(m,p) = \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m}$$



Mean and variance ('uncertainty')

$$\bar{m} = N \cdot p \approx N_{obs} \quad and \quad \sigma_m^2 = N \cdot p \cdot (1-p) \approx N_{obs}$$

$N_{obs} = \#$ *of "counts" observed*, $p \ll 1.0$

Statistical "error" of $N_{obs}: \sigma_m \approx \sqrt{N_{obs}}$

$$\frac{\sigma_m}{\bar{m}} = \frac{\sqrt{N \cdot p \cdot (1-p)}}{N \cdot p} \approx \frac{1}{\sqrt{N_{obs}}} \rightarrow$$ *more counts = smaller error*

Distributions P(m) approximates Gaussian very fast, already good for p=0.2-0.3

Observe Poisson → Gaussian

$$\lim_{\substack{p\to 1\\ N\to\infty}} P_{bin}(N,m,p) = \frac{1}{\sqrt{2\pi\sigma_m^2}} \cdot \exp\left\{-\frac{\left(x-\langle m\rangle\right)^2}{2\sigma_m^2}\right\}$$

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Poisson Probability Distribution

Results from binomial distribution in the limit of small p and large N (N·p > 0)

$$\lim_{\substack{p\to 0 \\ and\ N\to\infty}} P_{binomial}(N,m) = P_{Poisson}(\mu,m)$$

Probability for observing m events when average is <m> = μ

$$P_{Poisson}(\mu,m) = \frac{\mu^m \cdot e^{-\mu}}{m!}$$

m=0,1,2,…

μ = <m> = N·p and $\sigma^2 = \mu$

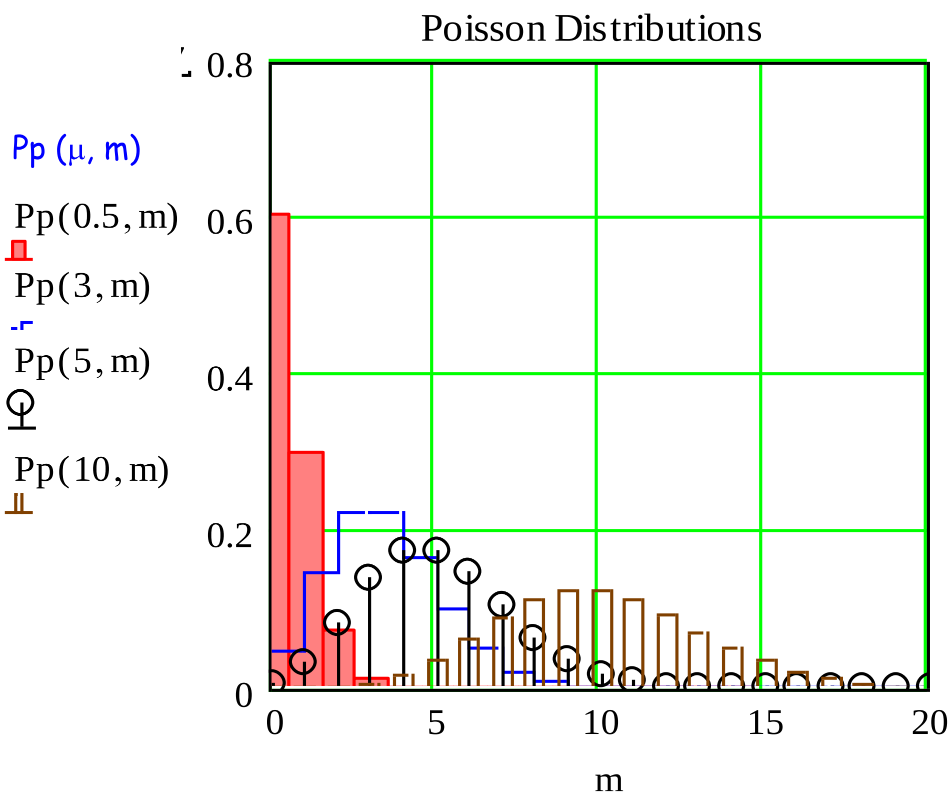
is the mean, the average number of successes in N trials.

Observe N counts (events) → → uncertainty is $\sigma = \sqrt{\mu}$

Unlike the binomial distribution, the Poisson distribution does not depend explicitly on p or N !

For increasing p (<1.0):

Poisson → Gaussian (Normal Distribution)



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Moments of Transition Probabilities

$$N : 0.25mg = 0.25mg \cdot \frac{6.022 \cdot 10^{23}}{236g} = 6.38 \cdot 10^{17}$$

$$p = \frac{\langle n \rangle}{N} = \frac{3.5946 \cdot 10^{4}\ \mathrm{min}^{-1}}{6.38 \cdot 10^{17}} = 5.6362 \cdot 10^{-14}\ \mathrm{min}^{-1}$$

Probability for decay (decay rate per nucleus):

$$p = \lambda = 5.6362 \cdot 10^{-14}\ \mathrm{min}^{-1}$$

$$\textit{corresponds to "half life"}\ t_{1/2} = 2.34 \cdot 10^{7}\, a$$

Small probability for process, but many trials ($n_0 = 6.38 \cdot 10^{17}$)

→ → → $0 < n_0 \cdot \lambda < \infty$

Statistical process follows a Poisson distribution: n="random"
Different statistical distributions: Binomial, Poisson, Gaussian

Radioactive Decay as Poisson Process

Useful when only a mean count rate is known: decay, background counts, or reaction.

^{137}Cs = unstable isotope, decays with
$t_{1/2}$ = 27 years → $p = \ln 2/27 = 0.026/a = \mathbf{8.2 \cdot 10^{-10} s^{-1}}$ → small

Sample of 1 μg: $N = 10^{15}$ nuclei (=trials for decay)
How many will decay?

$$\mu = N \cdot p = \mathbf{8.2 \cdot 10^{+5}\ s^{-1}}$$

Count rate estimate $dN/dt = (\mathbf{8.2 \cdot 10^{+5} \pm 905})\ \mathbf{s^{-1}}$
← estimated

Probability for m decays $P(\mu, m)$ =

$$P_{Poisson}(\mu, m) = \frac{\mu^m \cdot e^{-\mu}}{m!} = \frac{(8.52 \cdot 10^5)^m \cdot e^{-8.52 \cdot 10^5}}{m!}$$

Functions of Stochastic Variables

Random independent variables $N_1, N_2, \ldots, N_n$
corresponding variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$
Function $f(N_1, N_2, \ldots, N_n)$ of random variables: Uncertainty $\Delta N_i \rightarrow \Delta f(\{N_i\})$

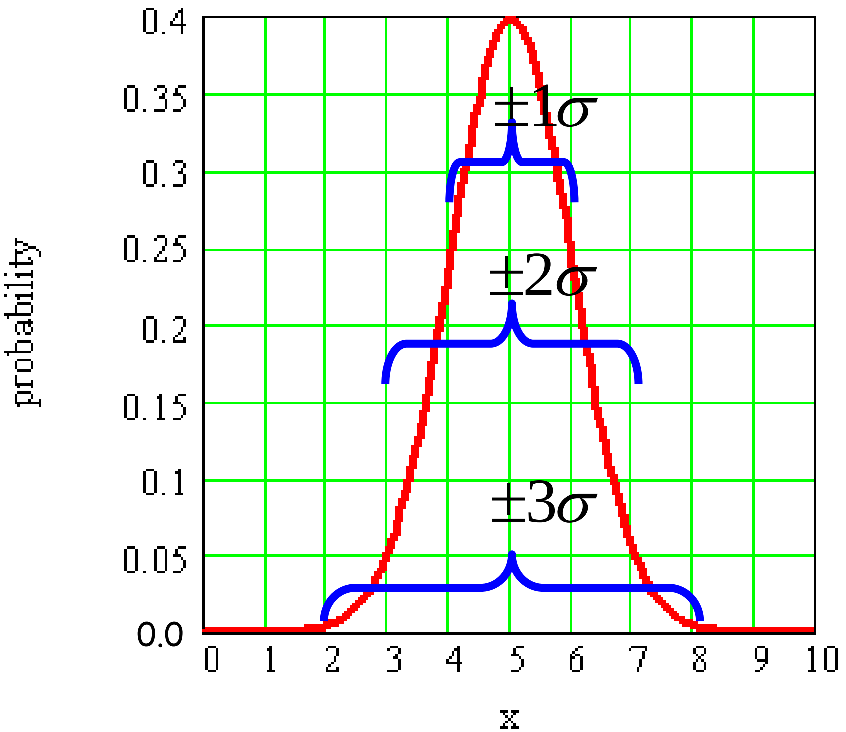
Gauss' law of error propagation:

$$\sigma_f \approx \left\{ \underbrace{\left(\frac{\partial f}{\partial N_1}\right)^2 \sigma_1^2}_{\left(\Delta f \,|_{N_2, N_3, ..}\right)^2} + \underbrace{\left(\frac{\partial f}{\partial N_2}\right)^2 \sigma_2^2}_{\left(\Delta f \,|_{N_1, N_3, ..}\right)^2} + \ldots + \underbrace{\left(\frac{\partial f}{\partial N_n}\right)^2 \sigma_n^2}_{\left(\Delta f \,|_{N_1, N_2, .., N_{n-1}}\right)^2} \right\}^{1/2}$$

Further terms if N_i are not independent (→ correlations, co-variance tensor)
Otherwise, individual independent component variances $(\Delta f)^2$ add.

Confidence Level

Measured Probability



Assume normally distributed observable x:

$$P(x) = \frac{1}{\sqrt{2\pi v_{pop}^2}} \cdot \exp\left\{ -\frac{\left(x - \langle x \rangle_{pop}\right)^2}{2v_{pop}^2} \right\}$$

Sample distribution with data set

→ observed average <x> and std. error σ approximate population. Confidence level CL (Central Confidence Interval):

$$P(|< x_{pop} > - < x >| < \delta) \approx$$

$$\approx \frac{2}{\sqrt{2\pi\sigma^2}} \cdot \int_{<x>}^{<x>+\delta} \exp\left\{ -\frac{\left(x - \langle x \rangle\right)^2}{2\sigma^2} \right\} dx = CL$$

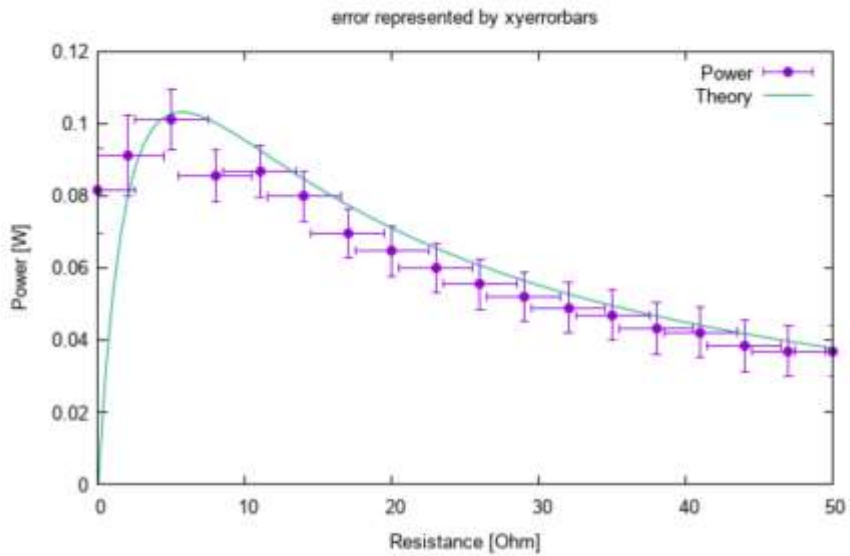
With (confidence level) CL probability, true value $\langle x_{pop} \rangle$ differs by less than $\delta = n\sigma$ from measured average.

$$CL(\delta = 1\sigma) = 68.3\% \quad CL(\delta = 2\sigma) = 95.4\%$$

$$CL(\delta = 3\sigma) = 99.7\%$$

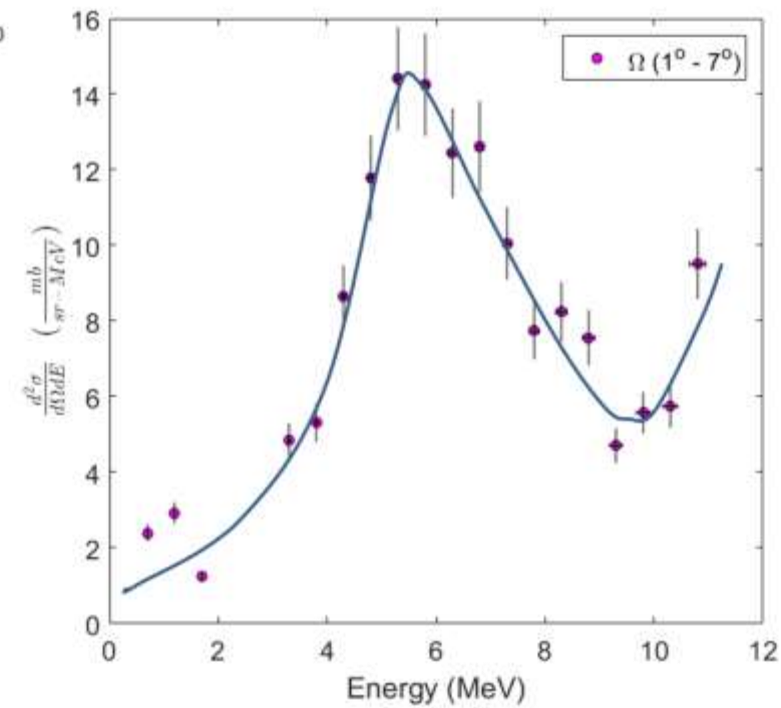
For very trustworthy exptl. results quote ±3σ error bars!

Stochastic Data Fits to Theory



Which of the 2 data plots show stochastic variation?

Experimental samples should scatter normally about the true theory function.



Setting Confidence Limits

Example: Search for rare particle decay with small decay rate λ, observe counts within time Δt. Decay probability law (survival prob.) $dP/dt \propto \exp\{-\lambda \cdot t\} \propto dP/d\lambda$.
Law is symmetric in λ and t : → $P(\lambda, t)$

Prob for no decay in $[0, \Delta t]$, particle has survived

$$\Delta P\left(0|\lambda\right) = \Delta t \cdot e^{-\lambda \cdot \Delta t} \cdot \Delta\lambda \quad with \quad \int_0^\infty \Delta t \cdot e^{-\lambda \cdot \Delta t} d\lambda = 1$$

Probability for survival during Δt at λ.

Example: no decays observed in ΔT

$$P(\lambda \le \lambda_0) = \int_0^{\lambda_0} \Delta T \cdot e^{-\lambda \cdot \Delta T} d\lambda = 1 - e^{-\lambda_0 \cdot \Delta T}$$

Prob for 1 decay for $\lambda = \lambda_0$

$$\lambda_0 = \frac{-1}{\Delta T} \cdot \ln[1 - P(\lambda \le \lambda_0)] = \frac{-1}{\Delta T} \cdot \ln[1 - CL\left(\lambda_0\right)] \ge 0$$

Lower limit for λ_0 = upper limit for λ

The higher the confidence level CL ($0 \le CL \le 1$), the larger the upper limit for λ for a given time ΔT inspected. Reduce limit by measuring for longer period or larger samples.

Binomial Distribution

Consider a stochastic process with two possible outcomes: yes/no, head/tail, success/failure,…, decay/remain intact

→ specific probability for success = *p* *and therefore for failure (1-p)*

Initiate this process **N** times → Question: What is the probability for ***m*** *successes?*

p^m = **probability for at** least *m* successes (observations) among *N* trials
$(1-p)^{N-m}$ = **probability for N-m failures** (survivals, not detected,…)

Probability for exactly *m* successes out of a total of *N* trials

$$P(m) = \binom{N}{m} \cdot p^m \cdot (1-p)^{N-m} \rightarrow \sum_m P(m) = 1$$

$$\binom{N}{m} = \frac{N!}{m!(N-m)!}$$

How many ways can *m* success events be 'chosen' out of N ? → Binomial coefficient

$$\langle m^\nu \rangle = \sum_{m=0}^{N} m^\nu \cdot P(m) = \sum_{m=0}^{N} m^\nu \cdot \binom{N}{m} p^m (1-p)^{N-m}$$

$$\text{Mean value } \mu = \langle m \rangle = N \cdot p; \qquad \text{Variance } \sigma_m^2 = N \cdot p \cdot (1-p)$$

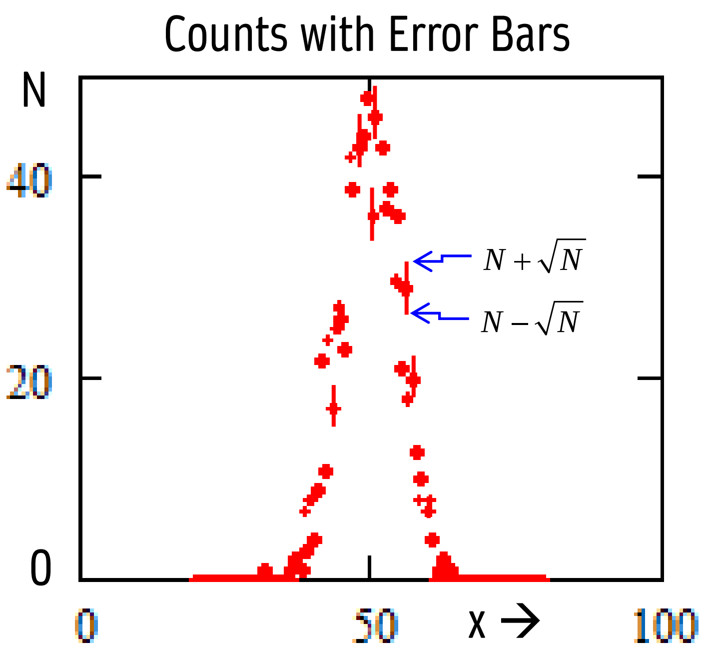
Poisson Probability Distribution

Limit of binomial distribution $Lim_{p\to 0, N\to\infty} P_{binomial}(N,m) = P_{Poisson}(\mu, m)$

Probability for observing m events when average is <m> = μ

$$P_{Poisson}(\mu,m) = \frac{\mu^m \cdot e^{-\mu}}{m!}$$

μ = <m> = N·p for $N \to \infty$
and $\sigma^2 = \mu$



For radioactive decays $[\Delta t^{-1}] \to p = \frac{A}{N}$

$p \ll 1 \to \sigma_m^2 \approx \langle m \rangle$ #counts

Observe N counts (events) →
→ statistical uncertainty is $\pm\sigma_N = \pm\sqrt{N}$

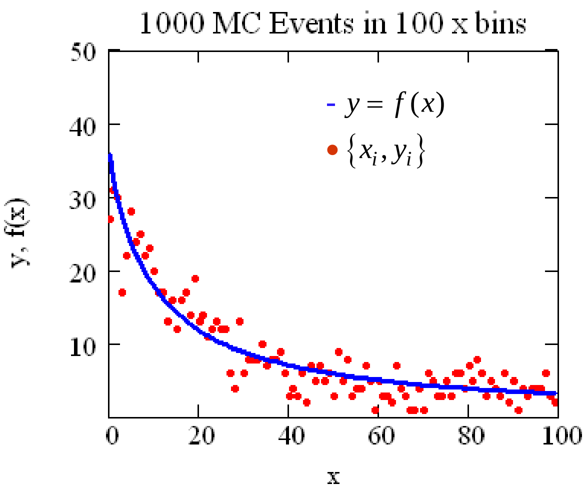
W. Udo Schröder, 2022

Curve Fitting to Data: Maximum Likelihood

Measurement of correlations between observables y and x: $\{x_i, y_i \mid i=1-N\}$
Hypothesis: $y(x) = f(c_1,\ldots,c_m; x)$. Only statistical errors. Parameters defining f: $\{c_1,\ldots,c_m\}$ $n_{dof}=N-m$ degrees of freedom for a "fit" of the data with f.

$$P_i(c_1,..,c_m;x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot \exp\left\{-\frac{\left(y_i - f(c_1,..,c_m;x_i)\right)^2}{2\sigma_i^2}\right\}$$

for every data point $\{y_i, x_i\}$, if f = true law



Maximize simultaneous probability for all points

$$P(c_1,..,c_m) = \prod_{i=1}^{N} P_i(c_1,..,c_m;x) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} \cdot \exp\left\{-\frac{(\Delta y_i)^2}{2\sigma_i^2}\right\}$$

$$\rightarrow \qquad -Ln\,P(c_1,..,c_m) \approx \sum_{i=1}^{N} \frac{(\Delta y_i)^2}{2\sigma_i^2} + const$$

$$\chi^2(c_1,..,c_m) := \sum_{i=1}^{N} \frac{(\Delta y_i)^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{\left(y_i - f(c_1,..,c_m;x_i)\right)^2}{\sigma_i^2}$$

When is the χ^2 as good as can be?

Minimize chi-squared by varying $\{c_1,\ldots,c_m\}$: $\partial\chi^2/\partial c_i = 0$

Minimizing χ^2

Example: linear fit $f(a,b;x) = a + b\cdot x$ to data set $\{x_i, y_i, \sigma_i\}$

Minimize:

$$\chi^2(a,b) := \sum_{i=1}^{N} \frac{(\Delta y_i)^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{(y_i - a - bx_i)^2}{\sigma_i^2}$$

$$0 = \frac{\partial}{\partial a}\chi^2(a,b) = \frac{\partial}{\partial a}\sum_{i=1}^{N} \frac{(y_i - a - bx_i)^2}{\sigma_i^2} = -\sum_{i=1}^{N} \frac{2(y_i - a - bx_i)}{\sigma_i^2}$$

$$0 = \frac{\partial}{\partial b}\chi^2(a,b) = -\sum_{i=1}^{N} \frac{2x_i(y_i - a - bx_i)}{\sigma_i^2}$$

Equivalent to solving system of linear equations

$$a\sum_{i=1}^{N}\frac{1}{\sigma_i^2} + b\sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} = \sum_{i=1}^{N}\frac{y_i}{\sigma_i^2}$$

$$a\sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} + b\sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2} = \sum_{i=1}^{N}\frac{x_i y_i}{\sigma_i^2}$$

$$ad_{11} + bd_{12} = c_1$$
$$ad_{21} + bd_{22} = c_2$$

$$D = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$$

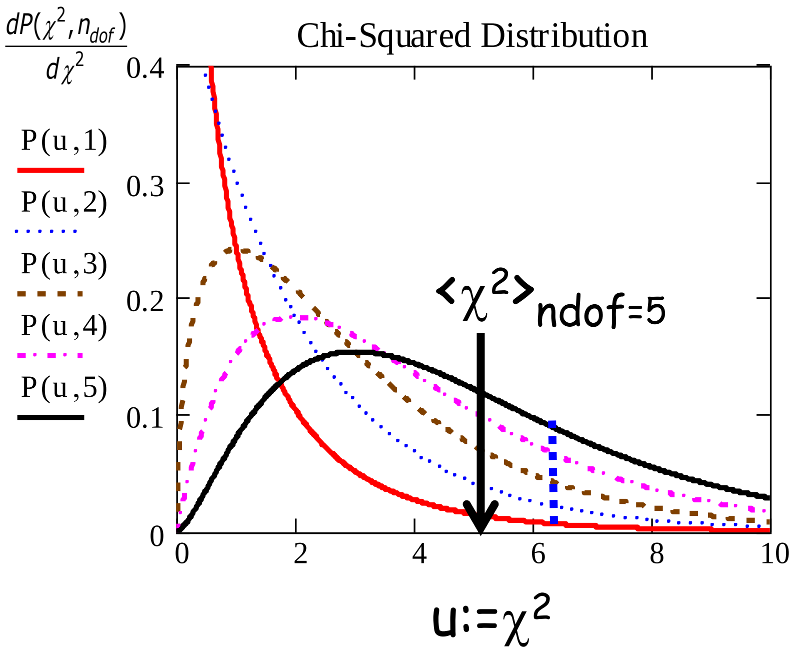
$$a = \frac{1}{D}\begin{vmatrix} c_1 & d_{12} \\ c_2 & d_{22} \end{vmatrix} \qquad b = \frac{1}{D}\begin{vmatrix} d_{11} & c_1 \\ d_{21} & c_2 \end{vmatrix}$$

$$\sigma_a^2 = \frac{1}{D}\sum\frac{x_i^2}{\sigma_i^2} \qquad \sigma_b^2 = \frac{1}{D}\sum\frac{1}{\sigma_i^2}$$

For more complex problems, solve by computer/numerical methods

Distribution of Chi-Squareds

Distribution of possible χ^2 for data sets that are distributed almost normally about a theoretical expectation (function) with n_{dof} degrees of freedom:



$$\frac{dP(\chi^2, n_{dof})}{d\chi^2} = \frac{\left(\chi^2\right)^{n_{dof}/2-1} e^{-\chi^2/2}}{2^{n_{dof}/2}\,\Gamma\left(n_{dof}/2\right)}$$

$\left\langle \chi^2 \right\rangle = n_{dof}$ $\quad \sigma^2_{\chi^2} = 2n_{dof}$ $\quad n_{dof} \gg 1$

$$\Gamma(n) = (n-1)! = \text{Stirling's formula}$$
$$= 2.507 e^{-n} n^{n-1/2}\left(1 + 0.0833/n\right)$$

$$P(\chi^2, n_{dof}) = \int_{\chi^2}^{\infty} \frac{dP(x, n_{dof})}{dx}\,dx$$

Reduced χ^2:

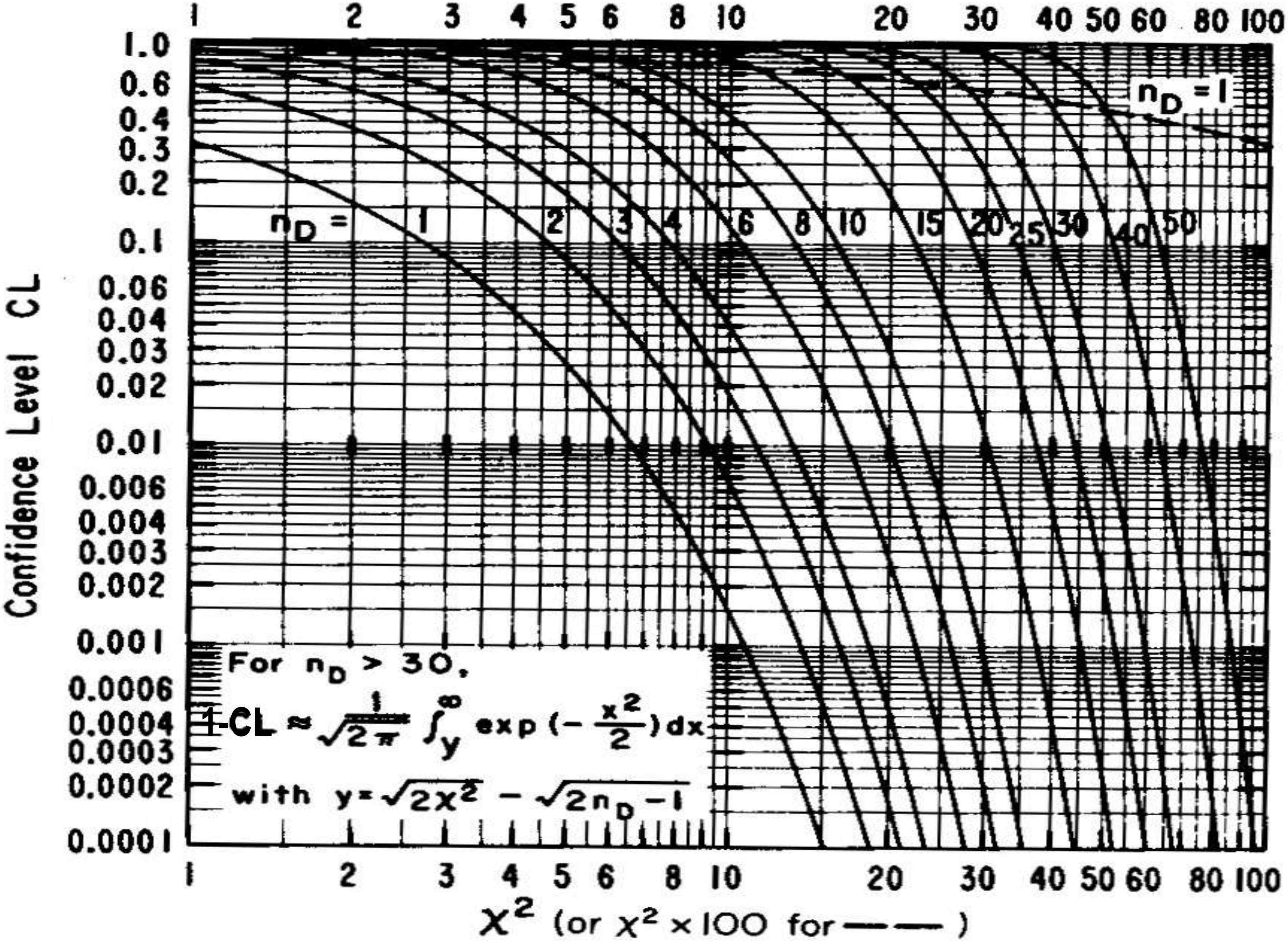
$$\chi_r^2 = \chi^2 / n_{dof} = \chi^2 / (N - m - 1)$$

For

$$0 \le \chi_r^2 < 1.5 \quad \rightarrow \quad \textit{Confidence} \ge 50\%$$

Should be $P \gtrsim 0.5$ for a "acceptable" fit

CL for χ^2-Distributions



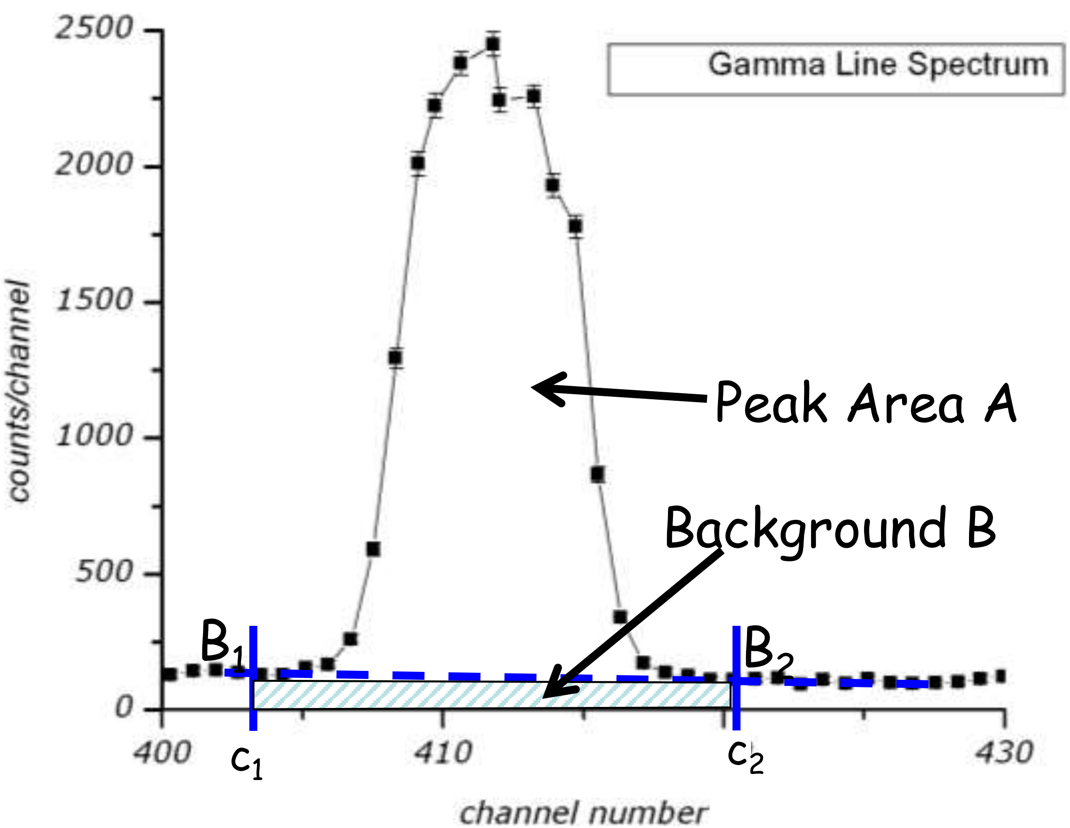
Example: Spectral Analysis

Adding or subtracting 2 Poisson distributed numbers N_1 and N_2:

Variances σ^2 always add

$$N := \left[N_1 \pm \sqrt{N_1}\right] \pm \left[N_2 \pm \sqrt{N_2}\right] \triangleq \left(N_1 \pm N_2\right) \pm \underbrace{\sqrt{N_1 + N_2}}_{\text{Std. dev } \sigma_{1\pm2}}$$

Std. dev σ_1 Std. dev σ_2 Std. dev $\sigma_{1\pm2}$



Analyze peak in range channels $c_1 - c_2$: beginning of background left and right of peak

$$n = c_1 - c_2 + 1.$$

Total area c_1 - c_2 → N_{12}

$N(c_1)=B_1$, $N(c_2)=B_2$,

Linear (≈constant) background

$B = n(B_1+B_2)/2$

$$\text{Peak area } A = \sum counts_i / ch$$

$$A = N_{12} - n \cdot \left(B_1 + B_2\right) / 2$$

Stat uncertainty std.dev.

$$\sigma_A = \sqrt{N_{12} + n \cdot \left(B_1 + B_2\right) / 2}$$